Resonance
Educating for better tomorrow

IIT-JEE 2012

PAPER - 2

PART - III : MATHEMATICS

Section I : Single Correct Answer Type

This section contains **8 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE is correct.**

41. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$ is

(A) $5x - 11y + z = 17$

(B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$

(D) $x - \sqrt{2}y = 1 - \sqrt{2}$

Sol. Ans. (A)

Equation of required plane

$$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$$

$\Rightarrow$ $(1 + \lambda)x + (2 - \lambda)y + (3 + \lambda)z - (2 + 3\lambda) = 0$

distance from point (3, 1, – 1)

$$= \left|\frac{3+3\lambda+2-\lambda-3-\lambda-2-3\lambda}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \left|\frac{-2\lambda}{\sqrt{3\lambda^2+4\lambda+14}}\right| = \frac{2}{\sqrt{3}}$$

$$\Rightarrow 3\lambda^2 = 3\lambda^2 + 4\lambda + 14$$

$$\Rightarrow \lambda = -\frac{7}{2}$$

equation of required plane

$$5x - 11y + z - 17 = 0$$

42. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}| = \sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a}+\vec{b}).(-7\hat{i}+2\hat{j}+3\hat{k})$ is

(A) 0 (B) 3 (C) 4 (D) 8

Sol. Ans. (C)

Let $\vec{c} = 2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{a}\times\vec{c} = \vec{c}\times\vec{b}$

$\Rightarrow (\vec{a}+\vec{b})\times\vec{c} = \vec{0}$

$\Rightarrow (\vec{a}+\vec{b}) \,||\, \vec{c}$

Let $(\vec{a}+\vec{b}) = \lambda\vec{c}$

$\Rightarrow |\vec{a}+\vec{b}| = |\lambda|\,|\vec{c}|$

$\Rightarrow \sqrt{29} = |\lambda| . \sqrt{29}$

$\Rightarrow \lambda = \pm 1$

$\therefore \vec{a}+\vec{b} = \pm(2\hat{i}+3\hat{j}+4\hat{k})$

Now $(\vec{a}+\vec{b}).(-7\hat{i}+2\hat{j}+3\hat{k}) = \pm(-14+6+12)$

$= \pm 4$

43. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and c = $\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Sol. Ans. (C)

a = 2 = QR

b = $\frac{7}{2}$ = PR

c = $\frac{5}{2}$ = PQ

s = $\frac{a+b+c}{2} = \frac{8}{4} = 4$

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)} = \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

44. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1,2,3,4,5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(A) $\frac{91}{216}$ (B) $\frac{108}{216}$ (C) $\frac{125}{216}$ (D) $\frac{127}{216}$

Sol. **Ans. (A)**

Favourable : D_4 shows a number and
only 1 of $D_1D_2D_3$ shows same number
or only 2 of $D_1D_2D_3$ shows same number
or all 3 of $D_1D_2D_3$ shows same number

Required Probability $= \frac{{}^6C_1({}^3C_1 \times 5 \times 5 + {}^3C_2 \times 5 + {}^3C_3)}{216 \times 6}$

$= \frac{6 \times (75 + 15 + 1)}{216 \times 6}$

$= \frac{6 \times 91}{216 \times 6}$

$= \frac{91}{216}$

45. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ell n\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is

(A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$

Sol. **Ans. (B)**

$$\int_{-\pi/2}^{\pi/2}\left(x^2 + \ell n\left(\frac{\pi+x}{\pi-x}\right)\right)\cos x\,dx = 2\int_0^{\pi/2} x^2\cos x\,dx + 0 \qquad \left(\because \ell n\left(\frac{\pi+x}{\pi-x}\right) \text{is an odd function}\right)$$

$$= 2\left[\left(x^2\sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x\sin x\,dx\right] = 2\left(\frac{\pi^2}{4} - 0\right) - 4\int_0^{\pi/2} x\sin x\,dx$$

$$= \frac{\pi^2}{2} - 4\left[\left(-x\cos x\right)_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right]$$

$$= \frac{\pi^2}{2} - 4$$

46. If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D) PX = – X

Sol. **Ans. (D)**

$P^T = 2P + I$

$\Rightarrow (P^T)^T = (2P + I)^T$

$\Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I$

$\Rightarrow 3P = -3I \quad \Rightarrow \quad P = -I$

$\Rightarrow PX = -IX = -X$

47. Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is

(A) 22 (B) 23 (C) 24 (D) 25

Sol. **Ans. (D)**

Corresponding A.P.

$\frac{1}{5}, \ldots\ldots\ldots\ldots \frac{1}{25}$ (20^{th} term)

$\frac{1}{25} = \frac{1}{5} + 19d \quad \Rightarrow \quad d = \frac{1}{19}\left(\frac{-4}{25}\right) = -\frac{4}{19 \times 25}$

$a_n < 0$

$\frac{1}{5} - \frac{4}{19 \times 25} \times (n-1) < 0$

$\frac{19 \times 5}{4} < n - 1$

$n > 24.75$

48. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a} - 1\right)x^2 + \left(\sqrt{1+a} - 1\right)x + \left(\sqrt[6]{1+a} - 1\right) = 0$ where $a > -1$.

Then $\lim\limits_{a \to 0^+} \alpha(a)$ and $\lim\limits_{a \to 0^+} \beta(a)$ are

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and –1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

Sol. **Ans. (B)**

$((1 + a)^{1/3} - 1)x^2 + ((a+1)^{1/2} - 1)x + ((a+1)^{1/6} - 1) = 0$

let $a + 1 = t^6$

$\therefore$ $(t^2 - 1)x^2 + (t^3 - 1)x + (t - 1) = 0$

$(t + 1)x^2 + (t^2 + t + 1)x + 1 = 0$

As $a \to 0$, $t \to 1$

$2x^2 + 3x + 1 = 0 \Rightarrow x = -1$ and $x = -\frac{1}{2}$

Section II : Paragraph Type

This section contains 6 multiple choice questions relating to three paragraphs with two questions on each paragraph. Each question has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

Paragraph for Question Nos. 49 to 50

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell n t\right) f(t)\, dt$ for all $x \in (1, \infty)$.

49. Which of the following is true ?

(A) g is increasing on $(1, \infty)$

(B) g is decreasing on $(1, \infty)$

(C) g is increasing on (1, 2) and decreasing on $(2, \infty)$

(D) g is decreasing on (1, 2) and increasing on $(2, \infty)$

Sol. Ans. (B)

$f(x) = (1 - x)^2 \sin^2 x + x^2 : x \in R$

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$$

$$\therefore\ g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x) \,.\, 1$$

let $\phi(x) = \frac{2(x-1)}{x+1} - \ln x$

$$\phi'(x) = \frac{2[(x+1)-(x-1).1]}{(x+1)^2} - \frac{1}{x} = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-x^2+2x-1}{x(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$$

$\therefore$ $\phi'(x) \le 0$

$\therefore$ for $x \in (1, \infty)$, $\phi(x) < 0$

$\therefore$ $g'(x) < 0$ for $x \in (1, \infty)$

50. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both P and Q are true (B) P is true and Q is false

(C) P is false and Q is true (D) both P and Q are false

Sol. **Ans. (C)**

$f(x) + 2x = (1 - x)^2 \sin^2 x + x^2 + 2x$

$\because \quad f(x) + 2x = 2(1 + x^2)$

$\Rightarrow \quad (1 - x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1 - x)^2 \sin^2 x = x^2 - 2x + 1 + 1$

$= (1 - x)^2 + 1$

$\Rightarrow \quad (1 - x)^2 \cos^2 x = -1$

which can never be possible

P is not true

$\Rightarrow$ Let $H(x) = 2f(x) + 1 - 2x(1 + x)$

$H(0) = 2f(0) + 1 - 0 = 1$

$H(1) = 2f(1) + 1 - 4 = -3$

$\Rightarrow$ so H(x) has a solution

so Q is true

Paragraph for Question Nos. 51 to 52

Let a_n denote the number of all n-digit positive integers formed by the digits 0,1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

51. Which of the following is correct ?

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{15}$ (D) $a_{17} = c_{17} + b_{16}$

Sol. **Ans. (A)**

1----------------- 1 # a_{n-1}

----------------- 1 0 # a_{n-2}

So A choice is correct

consider B choice $c_{17} \neq c_{16} + c_{15}$

$c_{15} \neq c_{14} + c_{13}$ is not true

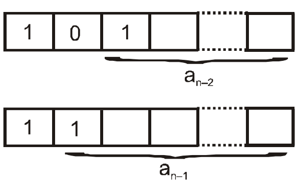
consider C choice $b_{17} \neq b_{16} + c_{16}$

$a_{16} \neq a_{15} + a_{14}$ is not true

consider D choice $a_{17} = c_{17} + b_{16}$

$a_{17} = a_{15} + a_{15}$ which is not true

Aliter



using the Recursion formula

$a_n = a_{n-1} + a_{n-2}$

Similarly $b_n = b_{n-1} + b_{n-2}$ and $c_n = c_{n-1} + c_{n-2}$ $\quad \forall\ n \geq 3$

and $\quad a_n = b_n + c_n \quad \forall\ n \geq 1$

so $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$..........

$b_1 = 1$, $b_2 = 1$, $b_3 = 2$, $b_4 = 3$, $b_5 = 5$, $b_6 = 8$ ....

$c_1 = 0$, $c_2 = 1$, $c_3 = 1$, $c_4 = 2$, $c_5 = 3$, $c_6 = 5$ ....

using this $b_{n-1} = c_n \ \forall\ n \geq 2$

52. The value of b_6 is

(A) 7 (B) 8 (C) 9 (D) 11

Sol. **Ans. (B)**

$b_6 = a_5$

$a_5 = \underline{1}$ - - - $\underline{1}$ $\quad \underline{1}$ - - - $\underline{0}$

${}^3C_0 + {}^3C_1 + 1 + {}^2C_1 + 1$

$1 + 3 + 1 + 2 + 1$

$4 + 4 = 8$

Paragraph for Question Nos. 53 to 54

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point P($\sqrt{3}$, 1). A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

53. A common tangent of the two circles is

(A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}\,y = 4$ (D) $x + 2\sqrt{2}\,y = 6$

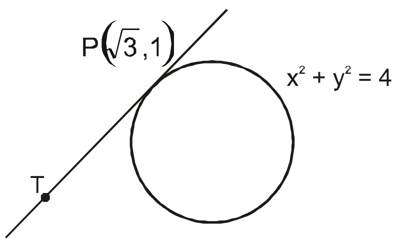
Ans. (D)

54. A possible equation of L is

(A) $x - \sqrt{3}\,y = 1$ (B) $x + \sqrt{3}\,y = 1$ (C) $x - \sqrt{3}\,y = -1$ (D) $x + \sqrt{3}\,y = 5$

Ans. (A)

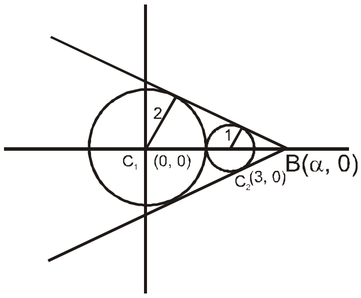
Sol. **Q.No. 53 to 54**



Equation of tangent at $(\sqrt{3}, 1)$

$\sqrt{3}x + y = 4$

53.



B divides C_1 C_2 in 2 : 1 externally

$\therefore$ B(6, 0)

Hence let equation of common tangent is

y – 0 = m(x – 6)

mx – y – 6m = 0

length of $\perp^r$ dropped from center (0, 0) = radius

$$\left|\frac{6m}{\sqrt{1+m^2}}\right| = 2 \Rightarrow m = \pm \frac{1}{2\sqrt{2}}$$

$\therefore$ equation is x + $2\sqrt{2}$ y = 6 or x – $2\sqrt{2}$ y = 6

54. Equation of L is

x – $y\sqrt{3}$ + c = 0

length of perpendicular dropped from centre = radius of circle

$$\therefore \left|\frac{3+C}{2}\right| = 1 \quad \Rightarrow C = –1, –5$$

$\therefore$ x – $\sqrt{3}$ y = 1 or x – $\sqrt{3}$ y = 5

Section III : Multiple Correct Answer(s) Type

This section contains **6 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONE or MORE are correct**.

55. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}$, $P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct ?

(A) $P(X \cup Y) = \frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P(X^C \cap Y) = \frac{1}{3}$

Sol. Ans. (AB)

$P(X/Y) = \frac{1}{2}$

$\frac{P(X \cap Y)}{P(Y)} = \frac{1}{2} \Rightarrow P(Y) = \frac{1}{3}$

$P(Y/X) = \frac{1}{3}$

$\frac{P(X \cap Y)}{P(X)} = \frac{1}{3} \Rightarrow P(X) = \frac{1}{2}$

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = \frac{2}{3}$ A is correct

$P(X \cap Y) = P(X) \cdot P(X) \Rightarrow$ X and Y are independent B is correct

$P(X^c \cap Y) = P(Y) - P(X \cap Y)$

$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ D is not correct

56. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then

(A) f has a local maximum at x = 2

(B) f is decreasing on (2, 3)

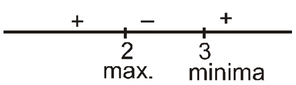
(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$

(D) f has a local minimum at x = 3

Sol. Ans. (ABCD)

$f(x) = \int_0^x e^{t^2} \cdot (t-2)(t-3)\,dt$

$f'(x) = 1 \cdot e^{x^2} \cdot (x-2)(x-3)$



(i) x = 2 is local maxima (ii) x = 3 is local minima (iii) It is decreasing in $x \in (2, 3)$

(iv) $f''(x) = e^{x^2} \cdot (x-2) + e^{x^2}(x-3) + 2x e^{x^2}(x-2)(x-3)$

$= e^{x^2} \cdot [x - 2 + x - 3 + 2x(x-2)(x-3)]$

$f''(x) = 0$

$f''(x) = e^{x^2}(2x^3 - 10x^2 + 14x - 5)$

$f''(0) < 0$ and $f''(1) > 0$

so $f''(c) = 0$ where $c \in (0, 1)$

57. For every integer n, let a_n and b_n be real numbers. Let function f : IR → IR be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}, \text{ for all integers n.}$$

If f is continuous, then which of the following hold(s) for all n ?

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

Sol. Ans. (BD)

$$\left.\begin{aligned} f(2n) &= a_n \\ f(2n^+) &= a_n \\ f(2n^-) &= b_n + 1 \end{aligned}\right\} \qquad \begin{aligned} a_n &= b_n + 1 \\ a_n - b_n &= 1 \end{aligned}$$

So B is correct

$$\left.\begin{aligned} f(2n+1) &= a_n \\ f((2n+1)^-) &= a_n \\ f((2n+1)^+) &= b_{n+1} - 1 \end{aligned}\right\} \qquad \begin{aligned} a_n &= b_{n+1} - 1 \\ a_n - b_{n+1} &= -1 \\ a_{n-1} - b_n &= -1 \end{aligned}$$

So D is correct

58. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

(A) y + 2z = –1 (B) y + z = –1 (C) y – z = –1 (D) y – 2z = –1

Sol. Ans. (BC)

For co-planer lines $[\vec{a} - \vec{c}\ \vec{b}\ \vec{d}] = 0$

$\vec{a} \equiv (1, -1, 0)$, $\vec{c} = (-1, -1, 0)$

$\vec{b} = 2\hat{i} + k\hat{j} + 2\hat{k}$ $\vec{d} = 5\hat{i} + 2\hat{j} + k\hat{k}$

Now $\begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \Rightarrow k = \pm 2$

$\vec{n}_1 = \vec{b}_1 \times \vec{d}_1 = 6\hat{j} - 6\hat{k}$ for k = 2

$\vec{n}_2 = \vec{b}_2 \times \vec{d}_2 = 14\hat{j} + 14\hat{k}$ for k = – 2

so the equation of planes are $(\vec{r} - \vec{a}).\vec{n}_1 = 0 \Rightarrow y - z = -1$ ...... (1)

$(\vec{r} - \vec{a}).\vec{n}_2 = 0 \Rightarrow y + z = -1$ ...... (2)

so answer is (B,C)

59. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

(A) –2 (B) –1 (C) 1 (D) 2

Sol. Ans. (AD)

Let $A = [a_{ij}]_{3\times 3}$

$$\text{adj } A = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$$

$|\text{adj } A| = 1(3-7) - 4(6-7) + 4(2-1) = 4$

$\Rightarrow |A|^{3-1} = 4$

$\Rightarrow |A|^2 = 4$

$\Rightarrow |A| = \pm 2$

60. Let $f : (-1, 1) \to \text{IR}$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value(s) of $f\left(\dfrac{1}{3}\right)$ is (are)

(A) $1 - \sqrt{\dfrac{3}{2}}$ (B) $1 + \sqrt{\dfrac{3}{2}}$ (C) $1 - \sqrt{\dfrac{2}{3}}$ (D) $1 + \sqrt{\dfrac{2}{3}}$

Sol. Ans. (AB)

$\cos 4\theta = \dfrac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \dfrac{1}{3} \Rightarrow \cos^2 2\theta = \dfrac{2}{3} \quad \Rightarrow \cos 2\theta = \pm\sqrt{\dfrac{2}{3}}$

Now $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta} = \dfrac{1+\cos 2\theta}{\cos 2\theta} = 1 + \dfrac{1}{\cos 2\theta}$

$\Rightarrow f\left(\dfrac{1}{3}\right) = 1 \pm \sqrt{\dfrac{3}{2}}$

NOTE : Since a functional mapping can't have two images for pre-image 1/3, so this is ambiguity in this question perhaps the answer can be A or B or AB or marks to all.