Regional Mathematical Olympiad-2013
Paper with Solution

Time : 3 hours **December 15, 2013**

Instructions :

- Calculators(in any form) and protactors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Maximum marks : 100
- Answer to each questions should start on a new page. Clearly indicate the question number.

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1. Given that a,b,c,d,e are real numbers such that

$$\left.\begin{array}{l} a+b+c+d+e = 8 \\ a^2+b^2+c^2+d^2+e^2 = 16 \end{array}\right]$$ (a,b,c d, e are all positive)

Determine the maximum value of e. **[17]**

Sol. **<u>Method- 1</u>** Given a, b, c, d, e $\in R^+$

$(a + b + c + d)^2 = (8-e)^2$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd = (8-e)^2$

$\Rightarrow 4(a^2 + b^2 + c^2 + d^2) \geq (8-e)^2$ [Using A.M. $\geq$ G.M.]

$\therefore 4(16 - e^2) \geq (8-e)^2$ [Given : $a^2 + b^2 + c^2 + d^2 = 16 - e^2$]

$\Rightarrow 5e^2 \leq 16e$

$\Rightarrow e \leq 16/5$

<u>Method- 2</u> For maximizing e, Take a = b = c = d

$4a + e = 8$ ......(1)

$4a^2 + e^2 = 16$ ......(2)

Solving (1) & (2)

$$e^2 + 4\left(\frac{8-e}{4}\right)^2 = 16$$

$\Rightarrow 4e^2 + (64 + e^2 - 16e)\ 64$

$\Rightarrow 5e^2 - 16e = 0, e > 0$

$\Rightarrow e = 16/5$

2. Let a,b,c be the sides of a triangle and A is its area. Prove that $a^2 + b^2 + c^2 \geq 4\sqrt{3}$ A. **[16]**

Sol. **Method- 1**

$a^2 + b^2 + c^2$

$= 4R^2 \sin^2 A + 4R^2 \sin^2 B + 4R^2 \sin^2 C$ [Using sine Rule]

$= 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C)$

We know that $\sin^2 A + \sin^2 B + \sin^2 C \geq 2\sqrt{3} \sin A . \sin B . \sin C$

Now, $a^2 + b^2 + c^2 \geq 4R^2 (2\sqrt{3} \sin A . \sin B . \sin C)$

$\Rightarrow a^2 + b^2 + c^2 \geq 2\sqrt{3}(2R \sin A)(2R \sin B) \sin C$

$a^2 + b^2 + c^2 \geq 4\sqrt{3}A$

Method- 2

By A.M –G.M. inequality

$$\frac{a^2 + b^2 + c^2}{3} \geq \sqrt[3]{a^2b^2c^2}$$

$\Rightarrow a^2 + b^2 + c^2 \geq 3(a^2b^2c^2)^{1/3}$

We know that

$$ab = \frac{2A}{\sin C}, \quad bc = \frac{2A}{\sin B}, \quad ca = \frac{2A}{\sin A}$$

$$\therefore \quad a^2b^2c^2 = \frac{8A^3}{\sin A \sin B \sin C}$$

Again we know that G.M $\leq$ A.M.

$$(\sin A \sin B \sin C)^{1/3} \leq \frac{\sin A + \sin B + \sin C}{3} \leq \frac{1}{3}\left(\frac{3\sqrt{3}}{2}\right)$$

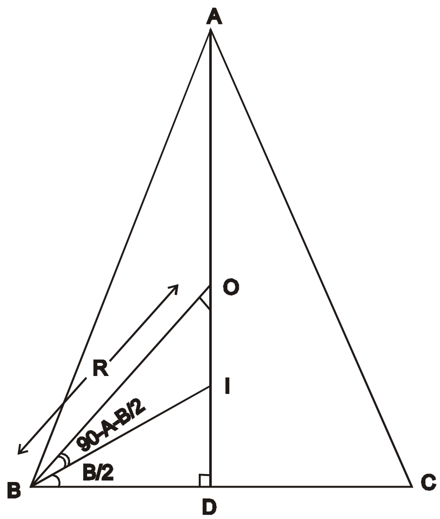
$$(\sin A \sin B \sin C)^3 \leq \frac{3\sqrt{3}}{8} \Rightarrow \frac{1}{\sin A \ \sin B \ \sin C} \geq \frac{8}{3\sqrt{3}}$$

$$\therefore \quad a^2b^2c^2 \geq \frac{64A^3}{3\sqrt{3}}$$

$a^2 + b^2 + c^2 \geq 3(a^2b^2c^2)^{1/3} \geq 4\sqrt{3}A$

3. Consider an isosceles triangle ABC. R is the radius of its circumscribed circle and r is the radius of its inscribed circle. Prove that the distance 'd' between the centres of these two circles is $\sqrt{R(R-2r)}$, when angle A $\le$ 60º and AB = AC. **[17]**

Sol.



Given, Isosceles triangle ΔABC. Let O & I are circumcentre and incentre respectively.

$\angle BAC \le 60^o$, O & I lies on altitude of BC.

$OB = R,\ IB = r\ \text{cosec}\frac{B}{2} = 4R\sin\frac{A}{2}\sin\frac{C}{2}$

$OI^2 = OB^2 + IB^2 - 2OB.\ IB\cos(\angle IBO)$ [Using cosine rule in ΔIBO]

$OI^2 = R^2 + 16R^2\sin^2\frac{A}{2}\sin^2\frac{C}{2} - 2R\left(4R\sin\frac{A}{2}\sin\frac{C}{2}\right)\cos\left(90^o - A - \frac{B}{2}\right)$

$OI^2 = R^2 + 16R^2\sin^2\frac{A}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{A}{2}\sin\frac{C}{2}\cos\left(\frac{C-A}{2}\right)$

$OI^2 = R^2 + 8R^2\sin\frac{A}{2}\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{C}{2} - \cos\left(\frac{C-A}{2}\right)\right]$

$OI^2 = R^2 + 8R^2\sin\frac{A}{2}\sin\frac{C}{2}\left(-\cos\left(\frac{A+C}{2}\right)\right)$

$OI^2 = R^2 + 8R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$OI^2 = R^2 - 2rR$ $\left[\text{Using } 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = r\right]$

4. Find all positive integers for which $(2^n - 1)$ in divisible by 7. **[17]**

Sol. **<u>Method- 1</u>**

$2^n - 1 \rightarrow$ divisible by 7

$2^n - 1 \equiv 0 \pmod 7$

$2^n \equiv 1 \pmod 7$

Clearly $2^{3k} \equiv 1 \pmod 7$, $2^{3k+1} \equiv 2 \pmod 7$, $2^{3k+2} \equiv 4 \pmod 7$

hence, n = 3k, $k \in N$

Method- 2

We observe the following set $\{2^1, 2^2, 2^3, \ldots\ldots\ldots\ldots 2^n\}$

Note that the numbers are of 3 types with respect to 3 namely 2^{3k} or 2^{3k+1} or 2^{3k+2} where k is positive integer.

Case- I

If number is of 2^{3k} type

$N = 2^{3k} = 8^k = (7 + 1)^k$

exapanding using binomial theorem

$N = {}^kC_0\, 7^k + {}^kC_1\, 7^{k-1} + \ldots\ldots\ldots\, {}^kC_{k-1}\, 7 + 1$

$= (7m + 1)$

$\therefore$ N – 1 is dividible by 7

$\therefore$ The whole case is the solution $\therefore$ n = 3k where $k \in N$ is the solution

Case- II

If number is of 2^{3k+1} type

$N = 2.8^k$

$= 2(7 + 1)^k$

similarly expanding we get

N = 7m + 2 type

$\therefore$ N – 1 is dividible by 7

$\therefore$ no solution

Case- III

If number is of 2^{3k+2} type

$N = 4.8^k$

$= 4\,(7 + 1)^k$

similearly expanding we get

N = 7m+4 types

$\therefore$ N – 1 not divisible by 7

$\therefore$ no solution

5. Solve $(x^2 + x-2)^3 + (2x^2 - x - 1)^3 = 27 (x^2 - 1)$ for real x. **[17]**

Sol. $(x^2 + x -2)^2 + (2x^2 - x - 1)^3 = 27 (x^2 - 1)$

$(x - 1)^3 (x + 1)^3 + (2x + 1)^3 (x -1)^3 = 27 (x + 1)^3 (x - 1)^3$

$(x - 1)^3 = 0$

$\therefore$ x = 1 is one of the solution.

Now let , $a = x + 2$, $b = 2x + 1$, $c = -3x- 3$

Hence, $a + b + c = x + 2 + 2x + 1 -3x -3 = 0$

If $a + b + c + 0$ then, $a^3 + b^3 + c^3 = 3abc$

So equation becomes $3(x + 2) (2x + 1) (-3x -3) = 0$

$(x + 2) (2x + 1) (-3x -3) = 0$

$x = -2, -½, -1$

Hence solution set is $\{-1, -½, -2, 1\}$

6. Determine all non negative integral pairs (x, y) for which $(xy - 7)^2 = x^2 + y^2$.

Sol. **<u>Method- 1</u>**

$(xy - 7)^2 = x^2 + y^2$

$\Rightarrow (x + y)^2 + xy (12-xy) = 49$

$\Rightarrow xy (12 - xy) = 0$ and $x + y = 7$

Case -I

$xy = 12$

$\Rightarrow x = 3, y = 4$ or $x = 4, y = 3$

Case -II $x y = 0$

$\Rightarrow$ $x = 0, y = 7$ or $x = 7, y = 0$

Required ordered pairs are : (3, 4) (4,3) (0, 7) (7, 0)

<u>Method- 2</u>

$(xy - 7)^2 = x^2 + y^2$

$\Rightarrow (x y - 6)^2 + 13 = (x+y)^2$

$\Rightarrow$ $(x + y -xy + 6) (x + y + xy - 6) = 13$

Case -I

$x + y + xy - 6 = 13$

$x + y - xy + 6 = 1$

On solving $(x, y) \equiv (4, 3) , (3, 4)$

Case -II

$x + y + xy - 6 = 1$

$x + y - xy + 6 = 13$

On solving $(x, y) \equiv (0, 7) , (7, 0)$

In all other cases negative solutions are obtained

hence solution set is (3,4) (4,3), (7, 0) (0, 7)