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RMO 2016

REGIONAL MATHEMATICAL OLYMPIAD 2016

TEST PAPER WITH SOLUTION & ANSWER KEY

REGION : GUJARAT | CENTRE : SURAT

Date: 09th October, 2016 | Duration: 3 Hours | Max. Marks: 102

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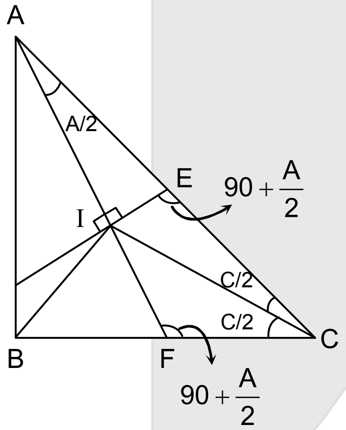
:: IMPORTANT INSTRUCTIONS ::

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ABC. Let AI extended intersect BC at F. Draw a line perpendicular to AI at I. Let it intersect AC at E. Prove than IE = IF.

Sol



Given : In ΔABC, $\angle$ABC = 90°, angle bisector of $\angle$BAC cut BC at F, I is incentre of ΔABC,

Line perpendicular to I through I cuts AC at E

To prove : IE = IF

Construction : Join IC

Prove : $\angle IAE = \frac{\angle A}{2}$ (AI is the angle bisector of $\angle$BAE)

$\angle IEC = 90 + \frac{A}{2}$ {$\angle$IEC is exterior angle of $\angle$AEI for ΔAEI}

$\angle IFC = 90 + \frac{A}{2}$ {$\angle$IFC is exterior angle of $\angle$AFB in ΔAFB}

Now in ΔIEC and ΔIFC

$\angle IEC = \angle IFC = 90 + \frac{A}{2}$ {from 1 and 2}

$\angle ECI = \angle FCI = \frac{C}{2}$ {CI is angle bisector of $\angle ECF$}

IC = IC (common)

so $\Delta IEC \cong \Delta IFC$ {AAS congruency criterion}

$\Rightarrow$ IE = IF {corresponding sides of congruent triangles}

2. Let a, b, c be positive real number such that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1$$

Prove that $abc \le \frac{1}{8}$.

Sol. $a(1+c)(1+a) + b(1+b)(1+a) + c(1+b)(1+c) = (1+a)(1+b)(1+c)$

$\Rightarrow (a^2 + b^2 + c^2) + (a^2c + b^2a + c^2b) = 1 + abc$ ....(1)

Now $\frac{a^2+b^2+c^2}{3} \ge (abc)^{2/3}$ (AM $\ge$ GM)

$\Rightarrow a^2 + b^2 + c^2 \ge 3(abc)^{2/3}$ ....(2)

$\frac{a^2c+b^2a+c^2b}{3} \ge (abc)$ (AM $\ge$ GM)

$\Rightarrow a^2c + b^2a + c^2b \ge 3abc$ ....(3)

add (2) and (3) we get

$(a^2 + b^2 + c^2) + (a^2c + b^2a + c^2b) \ge 3(abc)^{2/3} + 3abc$ ....(4)

$\Rightarrow 1 + abc \ge 3(abc)^{2/3} + 3(abc)$ (using (1) and (4)) ....(5)

Let $(abc)^{1/3} = t$

Now $1 + t^3 \ge 3t^2 + 3t^3$

$\Rightarrow (1+t)(1+t^2-t) \ge 3t^2(1+t) \Rightarrow (l+t)(3t^2-t^2+t-1) \le 0 \Rightarrow (1+t)^2(2t-1) \le 0$

$\Rightarrow t \le \frac{1}{2} \Rightarrow (abc)^{1/3} \le \frac{1}{2} \Rightarrow abc \le \frac{1}{8}$

3. For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

Ans. **17**

Sol. $n^3 = 8(S(n))^3 + 6n\,S(n) + 1$

$\Rightarrow \quad (2S(n))^3 + (-n)^3 + 1^3 - 3\,(2S(n))\,(-n)\,(1) = 0$

$\Rightarrow \quad (2S(n) + 1 - n)\left[\dfrac{(2S(n)+n)^2 + (1+n)^2 + (2S(n)-1)^2}{2}\right] = 0$

Because $(n + 1)^2$ is always positive so second factor is positive

$\Rightarrow \quad 2\,S(n) + 1 - n = 0$

$\Rightarrow \quad n = 2s(n) + 1$ ....(1)

Let $n = a_k\, a_{k-1}\, ,..........a_2a_1a_0$ where $a_k, a_{k-1}\, , ..., a_2\, , a_1\, , a_0$ represent digits

Now $(10^k\, a_k + 10^{k-1}\, a_{k-1} + ...+ 10a_1 + a_0) = 2\,(a_k + a_{k-1} + ...+ a_1 + a_0) + 1$

Because $2\,(a_k + a_{k-1} + ...+ a_1 + a_0) + 1 \geq 2\,(9\,(k + 1)) + 1$

{Equality holds when all digit are equal to 9}

so, $(10^k\, a_k + 10^{k-1}\, a_{k-1} + ...+ 10a_1 + a_0) \geq 2\,(9\,(k + 1)) + 1$

Which can holds only for $k = 0\, , 1$

Case-I $K = 0$

It means n is single digit number

Here $S(n) = n \quad \Rightarrow \quad n = 2\,n + 1$ {using (1)}

$\Rightarrow \quad n = -\,1 \quad \Rightarrow$ no natural number is possible

Case-II $K = 1$

It means n is two digit number.

Here $10a_1 + a_0 = 2\,(a_1 + a_0) + 1$ {using (1)}

$\Rightarrow \quad 8a_1 = a_0 + 1 \quad \Rightarrow \quad a_0 = 7$ and $a_1 = 1 \quad \Rightarrow$ n is 17

4. How many 6-digit natural numbers containing only the digits 1,2,3 are there in which 3 occurs exactly twice and the number is divisible by 9 ?

Ans. **0**

Sol. Let digits of six digit number are $a_5\, , a_4\, , a_3\, , a_2\, , a_1\, , a_0$ where $a_0\, , a_1\, , ...a_5 \in \{1\, , 2\, , 3\}$ in which two of them must be equal 3 and other four equals to either 1 or 2

Let $a_5 = a_4 = 3$ & $a_3\, , a_2\, , a_1\, , a_0 \in \{1, 2\}$

Now $a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 9k$

{If number is divisible by 9 then sum of digits is multiple of 9}

$\Rightarrow \quad a_3 + a_2 + a_1 + a_0 + 6 = 9k \quad \Rightarrow \quad a_3 + a_2 + a_1 + a_0 = 3\, , 12, 21\, , ...$

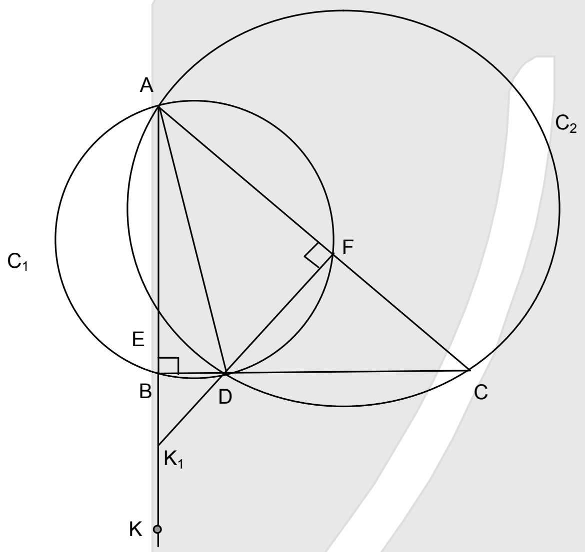
But minimum value of $a_0 + a_1 + a_2 + a_3 = 4$ (which is not possible) and maximum value of

$a_0 + a_1 + a_2 + a_3 = 12$ (Which occur when $a_0 = a_1 = a_2 = a_3 = 3$)

$\Rightarrow$ no number is possible

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Let the circumcircle of triangle ACD intersect AB again in E. and let the circumcircle of triangle ABD intersect AC again in F. Let K be the reflection of E in the line BC. Prove that FK = BC.

Sol.



Given : ABC is right angled triangle right angled at B, AD is angle bisector of $\angle BAC$ which cuts BC at D.

C_2 is circumcircle of ΔADC which cuts AB at E, C_1 is circumcircle of ΔABD which cuts AC at F.

K is reflection of E about line BD

To prove : FK = BC

Construction : Join FD and extend it so that its cuts extension of AB at K_1.

Proof : Points ABDF are concytic (given)

AD is diameter of circumcircle (C_1) of ΔABD ($\angle ABD$ is 90º)

$\angle AFD = 90º$ (because AD is diameter)

In ΔABD and ΔAFD

$\angle B = \angle F = 90º$, $\angle BAD = \angle FAD = \frac{\angle A}{2}$ (AD is angle bisector), AD is common

so $\Delta ABD \cong \Delta AFD$ (AAS congunecy theorem)

$\Rightarrow$ AB = AF and BD = FD .....(1)

$\angle AED = 180^\circ - \angle ACD$ (sum of opposite angles of cyclic quadrilateral AEDC is 180º)

$\angle BED = \angle ACD$ (Linear pair axiom)

$\angle EDB = 90^\circ - \angle ACD$ (Δ EBD is right triangle) ....(2)

$\angle FDC = 90^\circ - \angle ACD = \angle BDK_1$ ($\angle FDC$ and $\angle BDK_1$ are verticals opposite angle) ......(3)

Now In ΔEBD and ΔK_1BD

$\angle EDB = \angle K_1DB$ (from (2) and (3)), $\angle EBD = \angle K_1BD$ (both are 90º), BD is common

so, $\Delta EBD \cong \Delta K_1BD$ (AAS congruency theorem)

$\Rightarrow$ $EB = BK_1$ (corresponding sides of congruent triangle)

$\Rightarrow$ K_1 is reflection of E about line BD

$\Rightarrow$ K_1 is same as K

Now In ΔBDK and ΔFDC

$\angle BDK = \angle FDC$, $\angle KBD = \angle CFD = 90^\circ$, BD = DF (from (1))

$\Rightarrow \Delta BDK \cong \Delta FDC$

BD = DF ....(4)

DC = DK ....(5)

Add (4) and (5) we get BC = FK hence proved.

6. Show that the infinite arithmetic progression (1, 4, 7, 10,........) has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression one has $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

Sol. Consider three terms of this sequence a = 3p + 1, b = 3q + 1, c = 3r + 1 where p,q,r are in A.P.

So a,b,c will also be in A.P.

Now $\frac{abc}{a}, \frac{abc}{b}, \frac{abc}{c}$ $\Rightarrow$ bc, ca, ab are in H.P. and all three of bc, ca, ab are of the form 3n + 1 for some $n \in N$, hence there will be infinite triplets of $(bc, ca, ab) \equiv (a_1, a_2, a_3)$ which are in H.P.

If (a_1, a_2, a_3) and (b_1, b_2, b_3) are two such triplets.

Let $a_1 = (3q + 1)(3r + 1)$, $a_2 = (3r + 1)(3p + 1)$, $a_3 = (3p + 1)(3q + 1)$

where p + r = 2p

and $b_1 = (3m + 1)(3n + 1)$, $(3n + 1)(3\ell + 1)$, $(3\ell + 1)(3m + 1)$

where ℓ + n = 2m

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow \frac{(3q+1)(3r+1)}{(3m+1)(3n+1)} = \frac{(3r+1)(3p+1)}{(3n+1)(3\ell+1)}$

$\Rightarrow \frac{(3q+1)}{(3m+1)} = \frac{(3p+1)}{(3\ell+1)}$

Now for a choice of q, m, p, the value of ℓ is fixed, but ℓ can be chosen arbitrary for a given m.

Hence there are infinite triplets such that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. Thus completes the proof.

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