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2016

REGIONAL MATHEMATICAL OLYMPIAD
2016

TEST PAPER WITH SOLUTION & ANSWER KEY

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Date: 16th October, 2016 | Duration: 3 Hours | Max. Marks: 102

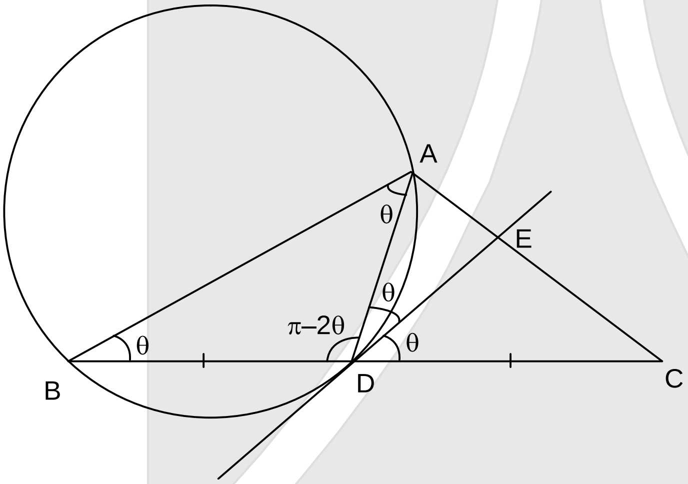
:: IMPORTANT INSTRUCTIONS ::

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a triangle and D be the mid-point of BC. Suppose the angle bisector of $\angle ADC$ is tangent to the circumcircle of triangle ABD at D. Prove that $\angle A = 90^o$.

Sol.



Given BD = DC, let the angle bisector of $\angle ADC$ meet AC at E,

Further assume $\angle CDE = \angle ADE = \theta$.

Since the angle bisector is tangent at D,

$\angle ABC = \theta$ (angle in alternate segment are equal)

Now $\angle ABD = \pi - 2\theta \Rightarrow \angle BAD = \theta \Rightarrow$ ABD is an isosceles triangle

So AD = BD = CD $\Rightarrow$ D is equidistant from vertices A,B,C

$\Rightarrow \Delta$ is circumcentre lies on triangle and is mid-point of BC $\Rightarrow \angle A = 90^o$

2. Let a,b,c be three distinct positive real number such that abc = 1.

Prove that $\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}\geq 3$

Sol. $-\left(\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(a-b)(b-c)(c-a)}\right)$

$= -\left(\frac{a^3b-a^3c+b^3c-b^3a+c^3a-c^3b}{(a-b)(b-c)(c-a)}\right)$

$= -\left(\frac{ab(a-b)(a+b)+c(b^3-a^3)+c^3(a-b)}{(a-b)(b-c)(c-a)}\right)$

$= -\left(\frac{ab(a+b)-c(a^2+b^2+ab)+c^3}{(b-c)(c-a)}\right)$

$= -\left(\frac{a^2b+ab^2-a^2c-b^2c-abc+c^3}{(b-c)(c-a)}\right)$

$= -\left(\frac{a^2(b-c)+ab(b-c)+c(c-b)(c+b)}{(b-c)(c-a)}\right)$

$=-\left(\frac{a^2+ab-c(c+b)}{(c-a)}\right)$

$= -\left(\frac{a^2+ab-c^2-bc}{(c-a)}\right)$

$= -\left(\frac{(a-c)(a+c)+b(a-c)}{(c-a)}\right) = a + b + c$

AM $\geq$ GM

$\frac{a+b+c}{3}\geq (abc)^{1/3}$

$a + b + c \geq 3$

3. Let a,b,c,d,e,f be positive integers such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$$

Suppose af – be = –1. show that $d \geq b + f$.

Sol. $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$

af – be = –1

Now to show that $d \geq b + f$

$ad + \lambda_1 = bc$ ....(1) λ_1 and $\lambda_2 \in I^+$

$cf + \lambda_2 = de$ ....(2)

af + 1 = be ....(3)

multiply the (1) equation by f

$bcf = afd + \lambda_1 f$

$b(de - \lambda_2) = d(be - 1) + \lambda_1 f$

$bde - \lambda_2 b = bde - d + \lambda_1 f$

$d = \lambda_2 b + \lambda_1 f$

$d \geq b + f$

4. There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n ?

Sol. Let there be 20 languages everybody speaks.

$P_1 = \{L_1, L_2 \text{...........} L_{20}\}$

$P_2 = \{L_1, L_2 \text{...........} L_{20}\}$

⋮

⋮

$P_{80} = \{L_1, L_2 \text{...........} L_{20}\}$

$P_{81} = \{L_2, L_3 \text{...........} L_{20}, L_{21}\}$

$P_{82} = \{L_1, L_3 \text{...........} L_{20}, L_{21}\}$

⋮

$P_{100} = \{L_1, L_2 \text{...........} L_{19}, L_{21}\}$

Now a group of 20 selected from $P_1 - P_{80}$ will be able to communicate, while a group of 20 from $P_{81} - P_{100}$ will have common L_{21}. If some are chosen from $P_1 - P_{80}$ and some from $P_{81} - P_{100}$, then at maximum 19 persons will be chosen from $P_{81} - P_{100}$, $\because$ at maximum 19 of L_1.....L_{20} languages will be lost and one will still remain common with $P_1 - P_{80}$ in set L_1, L_2.....L_{20}.

Now to understand why N < 20 in not possible.

Consider N = 19.

Assume $P_1 - P_{99}$ speaks L_1, L_2 ......L_{19}

So P_{100} speaks (L_{20}......L_{38})

Obviously in a group of 20 when P_{100} is selected they don't have common language.

P_1.........P_{98} speaks (L_1......L_{19})

P_{99} & P_{100} have 9 and 10 languages

$P_{99} \equiv \{L_1 \text{.......} L_9, L_{20} \text{......} L_{29}\}$

$P_{100} \equiv \{L_{10} \text{.......} L_{19}, L_{20} \text{......} L_{28}\}$

whenever P_{99} & P_{100} are chosen in group of 20 no common language will be there.

P_1 ....P_{97} speak (L_1......L_{19})

P_{98}, P_{99}, P_{100} will has these 19 languages : No language is common among there in these 19 languages.

$P_{98} \equiv \{L_1, L_2, L_{20}......L_{36}\}$

$P_{99} \equiv \{L_3, L_4, L_5, L_{20}......L_{35}\}$ In set of 20, when P_{98}, P_{99}, P_{100} are selected common language

$P_{100} \equiv \{L_6, L_7, L_{20}......L_{36}\}$

Likewise $P_1.....P_{81}$ speaks $(L_1......L_{19})$

$P_{82} = \{L_1, L_{20}......L_{37}\}$

$P_{83} = \{L_2, L_{20}......L_{37}\}$

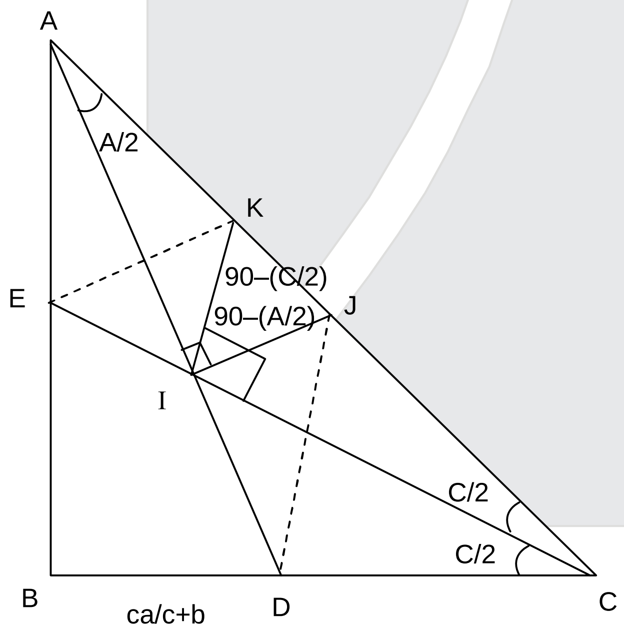
$\vdots$

$P_{100} = \{L_{19}, L_{20}......L_{37}\}$ Now when these 19 persons are chosen i -group of 20, common language will exist.

Hence Answer is 20

5. Let ABC be a right-angled triangle with $\angle B = 90^o$. Let I be the incentre of ABC. Extend AI and CI, let them intersect BC in D and AB in E respectively. Draw a line perpendicular to AI at I to meet AC in J, draw a line perpendicular to CI at I to meet AC in K. Suppose DJ = EK. Prove that BA = BC.

Sol.



Now given DJ = EK

$DJ^2 = EK^2$

$ID^2 + IJ^2 = EI^2 + IK^2$

In ΔIKC

$\tan\frac{C}{2} = \frac{IK}{IC}$

$IE^2 + IC^2 \tan^2\frac{C}{2} = (ID)^2 + (IA)^2 \tan^2\frac{A}{2} \Rightarrow \tan\frac{A}{2} = \frac{IJ}{IA}$ (since In ΔAIJ)

Now $IE = \frac{c}{a+b+c}$ EC (I divide CA in the ratio a + b : c)

$IC = \frac{a+b}{a+b+c}EC$

and $ID = \frac{a}{a+b+c}AD$ and $AI = \frac{b+c}{a+b+c}$ AD

Now $c^2(EC)^2 + (a+b)^2 EC^2\tan^2\frac{C}{2} = a^2(AD)^2 + (b+c)^2 AD^2\tan^2\frac{A}{2}$

Now in ΔEBC $\quad \tan\frac{C}{2} = \frac{EB}{a} = \frac{ac}{a+b}\cdot\frac{1}{a} \Rightarrow (a+b)\tan\frac{C}{2} = c$

In ΔABD $\quad \tan\frac{A}{2} = \frac{BD}{c} = \frac{ac}{(b+c)c} \Rightarrow (b+c)\tan\frac{A}{2} = a$

$\Rightarrow c^2(EC)^2 + c^2(EC)^2 = a^2(AD)^2 + a^2(AD)^2$

cEC = aAD $\Rightarrow \dfrac{ac}{\cos\frac{C}{2}} = \dfrac{ac}{\cos\frac{A}{2}} \qquad \left(\cos\frac{C}{2} = \frac{a}{EC}\right)$

$\Rightarrow \cos\frac{C}{2} = \cos\frac{A}{2} \Rightarrow \frac{C}{2} = \frac{A}{2} \Rightarrow C = A$

6. **(a)** Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

Sol. Consider the sequence

$$\frac{N!}{N}, \frac{N!}{N-1}, \frac{N!}{N-2}, \frac{N!}{N-3}, \ldots\ldots \frac{N!}{N-(N-1)}$$

$\Rightarrow \dfrac{N!}{N}, \dfrac{N!}{N-1}, \dfrac{N!}{N-2}, \ldots\ldots\ldots \dfrac{N!}{1}$ are in H.P.

Hence $\forall$ natural numbers N, we get a strictly increasing H.P. of N positive integers.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

Sol. Consider a harmonic progression whose first term, $T_1 = p$

second term $T_2 = q$ where $q > p$, $p, q \in N$ and all terms are positive integers.

Now for this H.P. $T_r = \dfrac{1}{\frac{1}{p} + (r-1)\left(\frac{1}{q} - \frac{1}{p}\right)}$

$\Rightarrow T_r = \dfrac{pq}{2q - p + r(p-q)}$

Now for $r > \dfrac{2q-p}{q-p}$, term of H.P. are negative,

Which is a contradiction

Hence the proof.

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