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REGIONAL MATHEMATICAL OLYMPIAD 2016

TEST PAPER WITH SOLUTION & ANSWER KEY

REGION: UTTAR PRADESH | CENTRE LUCKNOW

Date: 09th October, 2016 | Duration: 3 Hours | Max. Marks: 102

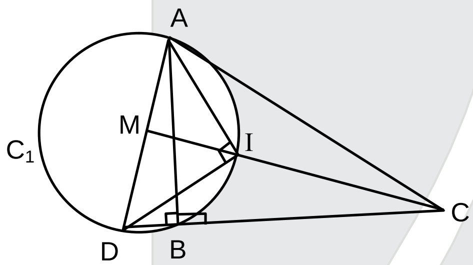
:: IMPORTANT INSTRUCTIONS ::

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that $ID = \sqrt{b(b-a)}$ where BC = a and CA = b

Sol.



Given : ABC is right angled triangle , right angled at B.

I is incentre of ΔABC

ID is perpendicular to IA (D lies on BC)

BC = a and CA = b ,

Construction : Let CI cuts AD at M

Join BI , Draw circumcircle (C_1) of ΔABD

To prove : CI is perpendicular to AD and ID = $\sqrt{b(b-a)}$

Proof : $\angle AID = \angle ABD = 90^o$

$\Rightarrow$ Points A, I, B, D are concyclic

$\Rightarrow$ A, I, B, D lies on C_1

$\Rightarrow \angle ABI = 45^o = \angle ADI$ .....(i)

(BI is angle bisector of $\angle ABC$. $\angle ADI$ and $\angle ABI$ are angle in same segment subtend by chord AI)

Now, $\angle AIM = \angle IAC + \angle ICA$ ($\angle AIM$ is exterior angle of $\angle AIC$ in ΔAIC)

$\Rightarrow \angle AIM = \frac{1}{2}(\angle BAC + \angle BCA) = \frac{1}{2}(90^o) = 45^o$ ....(ii)

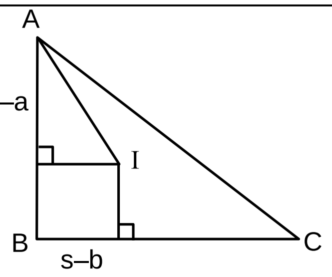
{AI and CI are angled bisector of $\angle A$ and $\angle C$

Now $\angle MID = \angle AID - \angle AIM = 90^o - 45^o = 45^o$ ....(iii)

Now in $\angle MDI$, $\angle DMI = 180^o - (\angle DIM + \angle MDI) = 180^o - (45^o + 45^o) = 90^o$

Hence proved

Now DI = AI {Δ AID is right angled isosceles triangle}



$\Rightarrow$ AI $= \sqrt{(s-a)^2 + (s-b)^2}$

$= \sqrt{\left(\frac{b+c-a}{2}\right)^2 + \left(\frac{a+c-b}{2}\right)^2}$ $\left\{\text{Let } AB = c \text{ and } s = \frac{a+b+c}{2}\right\}$

$= \sqrt{\frac{c^2 + (b-a)^2}{2}}$

$= \sqrt{\frac{b^2 - a^2 + b^2 + a^2 - 2ab}{2}}$

$= \sqrt{b^2 - ab}$ Hence Proved

2. Let a, b, c be positive real number such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$

Prove that $abc \le \frac{1}{8}$.

Sol. Let $\frac{a}{1+a} = x$, $\frac{b}{1+b} = y$, $\frac{c}{1+c} = z$ $(x, y, z > 0)$

Now we have given x + y + z = 1

And we have to prove $\left(\frac{x}{1-x}\right)\left(\frac{y}{1-y}\right)\left(\frac{z}{1-z}\right) \le \frac{1}{8}$

Proof: $\frac{x+y}{2} \ge (xy)^{\frac{1}{2}}$ …….. (i)

$\frac{y+z}{2} \ge (yz)^{\frac{1}{2}}$ …….. (ii)

$\frac{z+x}{2} \ge (xy)^{\frac{1}{2}}$ …….. (iii)

Multiply (i), (ii) and (iii) we get

$\frac{(x+y)(y+z)(z+x)}{8} \ge xyz$

$\frac{(1-z)(1-x)(1-y)}{8} \ge xyz$ $\{\therefore$ x+y+z$\}$

$\Rightarrow$ $\frac{xyz}{(1-x)(1-y)(1-z)} \le \frac{1}{8} \Rightarrow$ $abc \le \frac{1}{8}$

Hence Prove

3. For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

Ans. **50, 162, 392, 648**

Sol. Let numbe is $a_k\, a_{k-1}\, ...\, a_2a_1a_0$ where $a_0, a_1, ...\, a_k$ are digits.

It is given $(a_0 + 10a_1 + 100a_2 + ...+ 10^k a_k) = 2(a_0 + a_1 + a_2 + ... + a_k)^2$ .....(i)

Now $a_0 + a_1 + ...+ a_k \le 9(k + 1)$ {Equalty holds when all digit equals to 9}

$\Rightarrow 2(a_0 + a_1 + a_2 + ...+a_k)^2 \le 162\ (k +1\ )^2$ ....(ii)

From (i) and (ii) $\Rightarrow (a_0 + 10a_1 + ... + 10^k a_k) \le 162\ (k + 1)^2$

which can holds only for k = 0,1, 2 and 3

$\Rightarrow$ digits in the number are either 1 or 2 or 3 or 4

Case-I : When k = 3 (4 digit number)

If n is four digit number then $s(n) \le 36$

$\Rightarrow (S(n)^2 \le 1296 \Rightarrow \frac{n}{2} \le 1296 \Rightarrow n \le 2592$

$\Rightarrow$ max S(n) = 28 (when n = 1999) $\Rightarrow (S(n))^2 \le 784 \Rightarrow n \le 1568$

$\Rightarrow$ max. S(n) = 23 (when n = 1499) $\Rightarrow (S(n))^2 \le 529 \Rightarrow n \le 1058$

$\Rightarrow$ No four digit number is possible

Case-II : When k = 0, 1, 2

S(n)	$(S(n))^2 = n$	$2(S(n)^2 = n$	S(n)	$(S(n))^2 = n$	$2(S(n)^2 = n$
1	1	2	12	144	288
2	4	8	13	169	338
3	9	18	14	196	392 (accept)
4	16	32	15	225	450
5	25	50 (accept)	16	256	512
6	36	72	17	289	578
7	49	98	18	324	648 (accept)
8	64	128	19	361	722
9	81	162 (accept)	20	400	800
10	100	200	21	441	882
11	121	242	22	484	968

n can be 50, 162, 392, 648

4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

Ans. **281250**

Sol. Total number of 6 digit natural number formed (zero can be in I^{st} place) having exactly three odd digit and three even digit equal to ${}^6C_3\, 5^6$

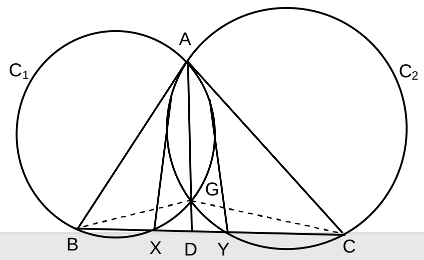
Total number of 5 digit natural number formed (zero can be in first place) having exactly 3 odd digit and 2 even digit equals to ${}^5C_3\, 5^5$

Total number of 6 digit natural number (zero cannot be in I^{st} place) having exactly three odd digit and three odd digit and three even digits equals to

${}^6C_3 5^6 - {}^5C_3 5^5 = (100 - 10)5^5 = 281250$

5. Let ABC be a triangle with centroid G. Let the circumcircle of triangle AGB intersect the line BC in X different from B; and the circumcircle of triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of triangle AXY.

Sol.



Given : Let D in mid point of side BC

Let C_1 is circumcircle of ΔABG which cuts BC at X

C_2 is circumcircle of ΔAGC which cuts BC at Y

Where G is centroid at ΔABC

To prove : G is centroid of ΔAXY

Proof : For circle C_1, chord AG and BX intersect at point D

So, (DX) (DB) = (DG) (DA).........(i)

For circle C_2, chord AG and CY intersect at point D

so (DY) (DC) = (DG) (DA) ....(ii)

From (i) and (ii)

(DG) (DA) = (DX) (DB) = (DY) (DC)

Because DB = DC so, DX = DY

$\Rightarrow$ D is mid point of XY $\Rightarrow$ AD is median from A to ΔAXY

$\Rightarrow$ G is centroid of ΔAXY

6. Let $\langle a_1, a_2, a_3, ...\rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

Sol. Let $a_1 = a$, $a_2 = a + d$, $a_3 = a + 2d$, ... and so on where a, d $\in$ N and A = {a_1, a_2, ... }

Let there exist a G.P. with first term equal to 'b' and integral common ratio equals to 'r' and whose terms are elements of set A.

Now let $b = a + k_1d$, $br = a + k_2d$ $\Rightarrow r = \dfrac{a+k_2d}{a+k_1d} \Rightarrow k_2 = \dfrac{a(r-1)}{d} + k_1r$

$\Rightarrow$ r can be taken of the form md + 1 (where m $\in$ N)

Now, we have to proof first that all terms of this G.P. belongs to set A.

$\Rightarrow$ we have to proof that br^n also belongs to set A $\forall$ n $\in$ N.

$\Rightarrow$ we have to proof that br^n is also of the form a + kd $\forall$ n $\in$ N where K $\in$ N

Now $br^n = b(md + 1)^n = b({}^nc_0(md)^n + {}^nc_1(md)^{n-1} + ... + {}^nc_{n-1}\, md + {}^nc_n)$

$= b\,(\lambda d + 1)$, where $\lambda \in$ N

$= b + b\lambda d = a + k_1d + b\lambda d = a + (k_1 + b\lambda)d$

$\Rightarrow br^n$ is also element of set A

Because m is variable and can take any natural value so infinite ratios (r) exist

$\Rightarrow$ Infinite G.P(s) exist of infinite length.

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