Regional Mathematical Olympiad-2015
क्षेत्रीय गणित ओलिंपियाड-2015

Time : 3 hours (समयः 3 घंटा) **December 06, 2015 (दिसम्बर 06, 2015)**

Instructions (निर्देश) :

- **Calculators (in any form) and protractors are not allowed.**
  किसी भी तरह के गुणक **(Calculators)** तथा चांदा के प्रयोग की अनुमति नहीं है।
- **Rulers and compasses are allowed.**
  पैमाना **(Rulers)** तथा परकार **(compasses)** के प्रयोग की अनुमति है।
- **Answer all the questions. All questions carry equal marks. Maximum marks : 102**
  सभी प्रश्नों के उत्तर दीजिये। सभी प्रश्नों के अंक समान हैं, अधिकतम् अंक : **102**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
  प्रत्येक प्रश्न का उत्तर नए पेज से प्रारंभ कीजिये। प्रश्न क्रमांक स्पष्ट रूप से इंगित कीजिये।

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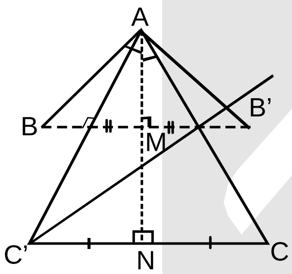
1. Let ABC be a triangle. Let B’ and C’ denote respectively the reflection of B and C in the internal angle bisector of $\angle$A. Show that the triangle ABC and AB’ C’ have the same incentre.

Sol.

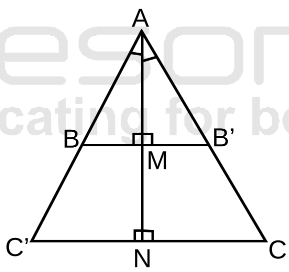


ΔABM $\cong$ ΔAB’M

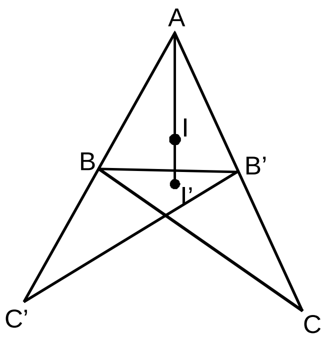
$\angle$ABM = $\angle$MAB’

$\therefore$ B’ lies over AC

same way C’ lie over AB when produced.



so angle bisector of ΔABC and ΔAB’C’ in same line AMN incentre of both lies on line AMN



as ABC $\cong$ AB'C'

$\therefore$ The distance of incentre I & I' is same from A

$\therefore$ AI = AI'

$\therefore$ II' = 0

$\therefore$ I & I' coincide

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numebrs $s \neq t$ such that P(s) = t and P(t) = s. Prove that b - st is a root of the equation $x^2 + ax + b - st = 0$.

Sol. $s^2 + as + b = t$ ..(1)

$t^2 + at + b = s$ ..(2)

Add (1) & (2)

$s(s+a) + t(a+t) + 2b = (s+t)$ ..(3)

subtract (1) from (2)

$(s^2 - t^2) + a(s - t) = (t - s)$

$(s - t)(a + s + t + 1) = 0$

$s - t = 0$ or $a + s + t + 1 = 0$

but $s \neq t$

$\therefore a + s + t + 1 = 0$

using (3) & (4)

$s(-t - 1) + t(-s - 1) + 2b = s + t$

$b - st = s + t$

$b - st = -1 - a$

$1 + a + b - st = 0$

$Q(x) = x^2 + ax + b - st$

if we put x = 1, $1 + a + b - st = Q(x)$

Q(x) = 0

so 1 is the root of $x^2 + ax + b - st = 0$

let other root α

$\alpha.1 = b - st$

$\alpha = b - st$

3. Find all integers a,b,c such that

$a^2 = bc + 1$, $b^2 = ca + 1$.

Sol. $a^2 = bc + 1$ ...(1)

$b^2 = ac + 1$ ...(2)

subtract (2) from (1)

$a^2 - b^2 = c(b - a)$

$(a - b)(a + b + c) = 0$

$a - b = 0$ or $a + b + c = 0$

I. If a – b = 0

a = b

put in (1)

$a^2 = ac + 1$

$a^2 - ac = 1$

$a(a - c) = 1$

a = 1 : a – c =1

a = 1, c = 0

if a = – 1, a – c = – 1

$\therefore$ (a, b c) = (1,1,0) (– 1, – 1, 0)

OR

II. a + b + c = 0

put a = –(b + c) in (1)

$(b + c)^2 = b + 1$

$b^2 + c^2 + bc = 1$

as abc are intiger

b = $\pm$1	c = $\pm$1	b = 1	b = – 1
c = 0	b = 0	c = – 1	c = 1
↓	↓	↓	↓
a = $\pm$1	a = $\pm$1	a = 0	a = 0

(–1, 1, 0), (1, – 1, 0), (– 1, 0, +1), (+1, 0, – 1), (0, 1, – 1), (0, – 1, 1) 6 cases

so total 8 cases

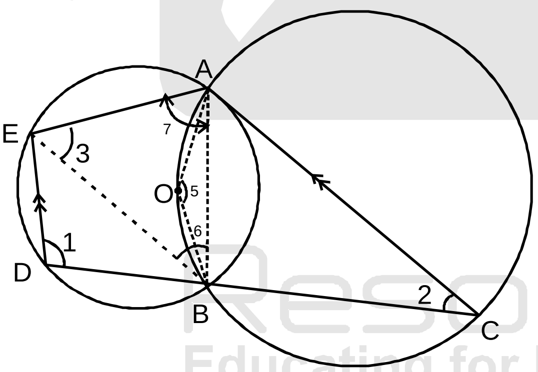
4. Suppose 32 objects are placed along a circle at equal distances, In how many ways can 3 objects be chosen from among them no two of the three chosen objects are adjacent not diametrically opposite ?

Sol.

Total way of selecting 3 points	${}^{32}C_3$	= 4960
3 pt together	32	= –32
Exactly 2p together	32×28	= – 896
Two points diametrically opposite and third is not adjacent to remaing two points	16×26	= – 416
		3616

5. Two circles Γ and Σ in the plane intersect at two distinct points A and B, and the centre of Σ lies on Γ. Let points C and D be on Γ and Σ respectively such that C,B and D are collinear. Let point E on Σ be such that DE is parallel to AC. Show that AE = AB.

Sol.



$\angle 2 = 180 - \angle 1$

$\angle 5 = 180 - \angle 2 = 180 - (180 - \angle 1)$

$\angle 5 = \angle 1$

$\angle 3 = \frac{\angle 5}{2} = \frac{\angle 1}{2}$

$\angle 7 = 180 - \angle 1$

$\therefore$ $\angle 6 = 180 - (\angle 3 + \angle 7)$

$= 180 - \left(\frac{\angle 1}{2} + 180 - \angle 1\right)$

$= 180 - 180 + \frac{\angle 1}{2}$

$\angle 6 = \frac{\angle 1}{2}$

$\therefore \quad \angle 6 = \angle 3 = \frac{\angle 1}{2}$

$\therefore \quad \angle 6 = \angle 3 = \frac{\angle 1}{2}$

$\therefore \quad$ A E = AB.

6. Find all real numbers a such that 4 < a < 5 and a(a – 3{a}) is an integer (Here {a} denotes the fractional part of a. For example {1.5} = 0.5 ; {–3.4} = 0.6).

Sol. $4 < a < 5$

Let $a = 4 + f$

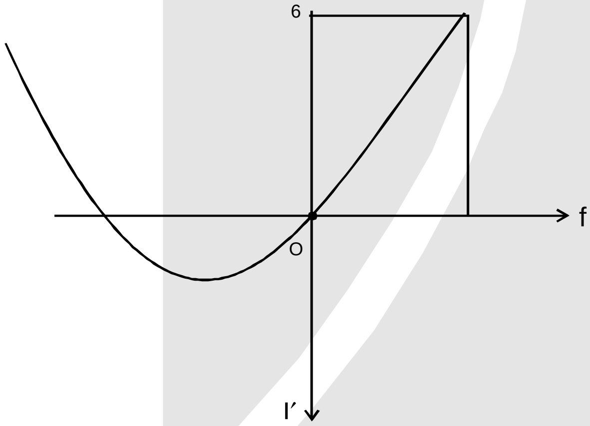
$a(a - 3\{a\}) = I$

$(4 + f)(4 + f - 3f) = I$

$(4 + f)(4 - 2f) = I$

$16 - 4f - 2f^2 = I$

$2f^2 + 4f = I'$ (intger)



as $0 < f < 1$

$\therefore \quad 0 < I' < 6$

$I' = 1, 2, 3, 4, 5$

$4f + 2f^2 = 1, 2, 3, 4, 5.$

we get 5 different value of f

If $4f + 2f^2 = 1$

$f = \frac{-2+\sqrt{6}}{2}$

$a = 4 + f = \frac{6+\sqrt{6}}{2}$

in the same way we can find other value of a

$a = \frac{6+\sqrt{8}}{2}, \frac{6+\sqrt{10}}{2}, \frac{6+\sqrt{12}}{2}, \frac{6+\sqrt{14}}{2}$

so we get 5 solution.