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REGIONAL MATHEMATICS OLYMPIAD – 2015

(Mumbai Region)

Date : 06-12-2015 Time: 4 Hrs. Max. Marks : 100

Instructions:

- There are eight questions in this question paper. Answer all questions
- Each of the questions 1,2,3 carries 10 points. Each of the questions 4,5,6,7,8 carries 14 points
- Use of protractors, mobile phone is forbidden
- Time allotted : 4 hours

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1. Let ABCD be a convex quadrilateral with AB = a, BC = b, CD = c and DA = d. Suppose

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$$

and the area of ABCD is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

Sol. Given AB = a, BC = b, CD = c and DA = d and $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$

$\Rightarrow$ $2\left(a^2 + b^2 + c^2 + d^2 - ab - bc - cd - da\right) = 0$

$\Rightarrow$ $(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 = 0$

$\Rightarrow$ So $a - b = b - c = c - d = d - a = 0$

$\Rightarrow$ a = b = c = d

Hence ABCD a Rhombus

Area of Rhombus = 60 sq. units

$$\frac{1}{2}d_1 d_2 = 60$$

$$\frac{1}{2} \times 30 \times d_2 = 60$$

$$d_2 = 4$$

So length of the other diagonal is 4 units

2. Determine the number of 3-digit numbers in base 10 having at least one 5 and at most one 3.

Sol.

No of 5	No of 3	other than 5 & 3
3	0	0
2	0	1
2	1	0
1	0	2
1	1	1

First case : Only one Number (555)

2^{nd} case : $8\times\frac{3!}{2!}-1$ (when 0 at 1^{st} position) = 23

3^{rd} case : $\frac{3!}{2!}=3$

4^{th} case :

5	8 ways	8 ways = 64	
7 ways	5	8 ways = 56	$\Rightarrow 176$
7 ways	8 ways	5 = 56	

5^{th} case : When 5 & 3 occupy 1^{st} & 2^{nd} place = 8 × 2 = 16

When 5 & 3 occupy 1^{st} & 3^{rd} place = 8 × 2 = 16

If 5 & 3 occupy 2^{nd} & 3^{rd} place

7 ways	5	8	$=14$
7 ways	8	5	

Total ways $=1+23+3+176+46=249$

3. Let P(x) be a polynomial whose coefficients are positive integers. If P(n) divides $P(P(n)-2015)$ for every natural number n, prove that $P(-2015)=0$.

Sol. Let P(x) be a polynomial of type

$$P(x)=a_0x^n+a_1x^{n-1}+.....+a_{n-1}x^0+a_n$$

$$P(P(x)-2015)=a_0(P(x)-2015)^n+a_1(P(x)-2015)^{n-1}\ ....+a_{n-1}(P(x)-2015)+a_n$$

$$P(P(x)-2015)=P(x)[\ \lambda\ ]+a_0(-2015)^n+a_1(2015)^{n-1}+.....+a_{n-1}(-2015)+a_n$$

$$P(P(x)-2015)=P(x)\lambda+P(-2015)$$

If it is divisible by P(x) for every x then $P(-2015)=0$

4. Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bac)_{10}$ and $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10)

Sol. Let $100a + 10b + c = A$

$a + 100b + 10x = Ar$

$10a + b + 100\,c = Ar^2$

Add all $111\,(a + b + c) = A\,(1 + r + r^2)$

$37 \times 3\,(a + b + c) = A\,(1 + r + r^2)$

A is 3 digit number & $3 \le a + b + c \le 27$

$\therefore$ $A = 37\,(a + b + c)$ & $r^2 + r + 1 = 3$

$\therefore$ $r = 1$ or $r = -2$

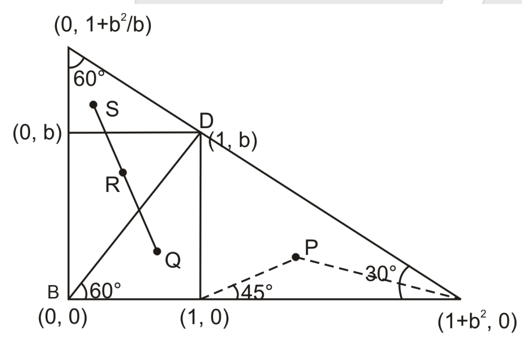
$r = 1 \Rightarrow$ All numbers are same

$\therefore$ $a = b = c$

$\therefore$ 9 number

5. Let ABC be a right angled triangle with $\angle B = 90°$ and let BD be the altitude from B on to AC. Draw DE $\perp$ AB and DF $\perp$ BC. Let P, Q, R and S be respectively the incenter of triangle DFC, DBF, DEB and DAE. Suppose S, R, Q are collinear. Prove that P, Q, R, D be on a circle.

Sol.



Q $\left(\dfrac{\sqrt{1+b^2}+1}{1+b+\sqrt{1+b^2}}, \dfrac{b}{1+b+\sqrt{1+b^2}}\right)$

R $\left(\dfrac{b+1+b^2+\sqrt{b^2+1}}{1+b+\sqrt{1+b^2}}, \dfrac{b^2}{1+b+\sqrt{1+b^2}}\right)$

(Similarly finding P in terms of B) incenter = $\left(\dfrac{aX_1+bX_2+cX_3}{a+b+c}, \dfrac{aY_1+bY_2+cY_3}{a+b+c}\right)$

$m_R = \dfrac{(b^2-b)}{b^2+b} = \dfrac{b-1}{b+1}$

S $\left(\dfrac{1}{1+b+\sqrt{b^2+1}}, \dfrac{b\sqrt{b^2+1}+1+b^2+b}{1+b+\sqrt{b^2+1}}\right)$

$m_{SR} = \dfrac{1+b+b\sqrt{b^2+1}}{-\sqrt{1+b^2}}$

$m_{SQ} = \dfrac{1+b^2+b\sqrt{b^2+1}}{-\sqrt{1+b^2}} = \dfrac{b+\sqrt{1+b^2}}{-1} = -b-\sqrt{1+b^2}$

$m_{SQ} = m_{SR}$ as Q, R and S are collinear

$\therefore$ $b = \sqrt{3}$

Hence

$m_{PR} = -\frac{1}{\sqrt{3}}$

$m_{RQ} = -\left(\sqrt{3}+2\right)$

$m_{PD} = -\sqrt{3}$

$m_{QD} = \left(2+\sqrt{3}\right)$

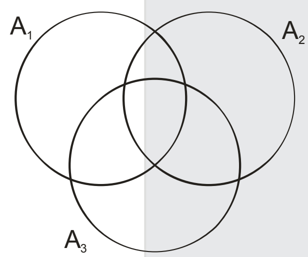
Hence $\angle QRP = \angle QDP = 45°$

6. Let $S = \{1, 2, ..., n\}$ and let T be the set of all ordered triples of subsets of S, Say (A_1, A_2, A_3), such that $A_1 \cup A_2 \cup A_3 = S$. Determine in term of n,

$$\sum_{(A_1, A_2, A_3)\varepsilon T} |A_1 \cap A_2 \cap A_3|$$

Where |X| denotes the number of elements in the set X, (For example, if S = {1,2,3} and $A_1 = \{1,2\}, A_2\{2,3\}, A_3 = \{3\}$ then one the element of T is ({1, 2}, {2,3}, {3}).

Sol.



1 element common : ${}^nC_1$ ways to select, remaining n – 1 elements to be distributed in 6 ways

$\therefore \quad {}^nC_1 6^{n-1}$ ways

Similarly 2 element common ${}^nC_2 6^{n-2}(2)$

(cardinality is 2)

$$\therefore \quad \sum_{r=1}^{n} {}^nC_r\, r\, 6^{n-r} = \sum_{r=1}^{n} n\, {}^{n-1}C_{r-1} 6^{n-r}$$

$= n\,7^{n-1}$

7. Let x, y, z be real number such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that $(1+x)(1+y)(1+z) \le 4 + 4xyz$.

Sol. $1 + x + y + 2 + xy + yz + xyz - 4 - 4 \times xy \le 0$

$x + y + z + xy + yz + zx - 3 - 3xyz \le 0$

$$x + y + z + xy + yz + zx - 3\left(1 + \frac{x^2 + y^2 + z^2 - 1}{2}\right)$$

$$x + y + z + xy + yz + zx - 3\left(\frac{x^2 + y^2 + z^2 + 1}{2}\right)$$

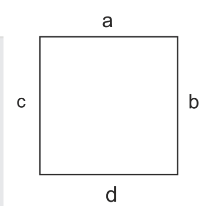
$$-\frac{1}{2}\left(3x^2+3y^2+3z^2+3-2x-2y-2z-2xy-2yz-2zx\right)$$

$$-\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2+(x-1)^2+(y-1)^2+(z-1)^2\right]$$

$$\Rightarrow \quad \le 0$$

8. The length of each side of a convex quadrilateral ABCD is a positive integer. If the sum of the lengths of any three sides is divisible by the length of the remaining side then prove that some two sides of the quadrilateral have the same length.

Sol. $a+b+c=k_1d$



$c+d+a=k_2b$

$a+b+d=k_3c$

$c+b+d=k_4a$

P = d (k_1 + 1) = b (k_2 + 1) = c (k_3 + 1) = a (k_4 + 1)

$$\frac{d}{b}=\frac{k_2+1}{k_1+1}$$

$$\frac{b}{c}=\frac{k_3+1}{k_2+1}$$

$$\frac{b}{a}=\frac{k_4+1}{k_2+1}$$

$$\frac{d}{c}=\frac{k_3+1}{k_1+1}$$

$$\frac{b}{c}\times\frac{c}{d} \qquad \frac{d+k_3+1}{c+k_1+1}=\frac{d}{c}$$

$$\frac{a+\dfrac{a+b+d}{c}+1}{c+\dfrac{a+b+c}{d}+1}$$

$$\frac{d(ac+a+b+d+c)}{c(ac+a+b+c+d)}=\frac{d}{c}$$

$$ac+a+b+d=ac+a+b+c$$

$$c=d$$

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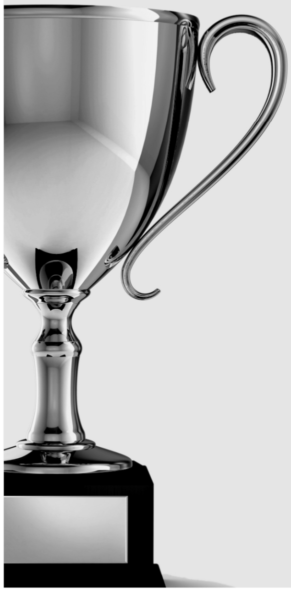

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