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REGIONAL MATHEMATICS OLYMPIAD – 2015

Centre - Delhi

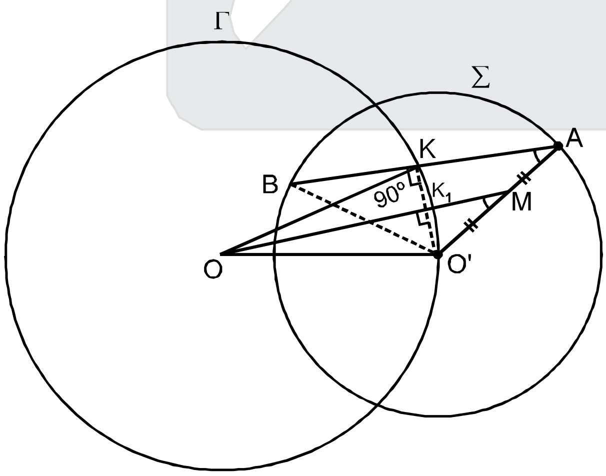
Time: 3 Hours **December 06, 2015**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Two circles Γ and Σ, with centres O and O′, respectively, are such that O′ lies on Γ. Let A be a point on Σ and M the midpoint of the segment AO′. If B is a point on Σ different from A such that AB is parallel to OM, show that the midpoint of AB lies on Γ.

Sol.



To prove O′ K $\perp$ AB as O′A = O′B (radii of circle Σ)

Also OK = OO′ (radii of circle Γ)

$\Delta O'MK_1$ and $\Delta O'AK$ are similar

Triangles $\Rightarrow K_1O' = KK_1 \Rightarrow \angle O'K_1O = 90^\circ$

Which implies $\angle O'KB = 90^\circ$

or KA = KB

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a and b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$ is an arithmetic progression of integers. Prove that a and b are integers.

Sol. Given $[P(-1)]^2 = (b - a + 1)^2 \in I$

$(P(0))^2 = b^2 \in I$

$(P(1))^2 = (b + a + 1)^2 \in I$

$\because$ $[P(-1)]^2, (P(0))^2, (P(1))^2$ are in AP

$\Rightarrow$ $2b^2 = (b - a + 1)^2 + (b + a + 1)^2$

$\Rightarrow$ $a^2 + 2b + 1 = 0$

$a^2 + b^2 + 1 - 2a + 2b - 2ab = I_1$ ------- (1)

$a^2 + b^2 + 1 + 2a + 2b + 2ab = I_2$ ------- (2)

(2) – (1) $\quad 4a(1 + b) = I$ ------- (3)

$b^2 \in I$

Case-I $b \in Q$

So if b is any rational number then it can be integer only as its square is integer.

$\Rightarrow$ $\boxed{b \in I}$

$\because$ $(a + b + 1)^2$ is integer (i) $a \in Q \Rightarrow a \in$ Integer

(ii) $a \in Q^c \Rightarrow$ then $(b - a + 1)^2$ can not be an integer

Case-II $b \notin Q$

b^2 is integer

$\because$ $a^2 + 2b + 1 = 0$ (from A.P. condition)

$\Rightarrow$ $2b = -(1 + a^2)$

$\Rightarrow$ b is negative

Let $b^2 = m$; $m \in I$

$\Rightarrow$ $b = -\sqrt{m}$

$a^2 = 2\sqrt{m} - 1$

From Eqn (3) $4a(1 + b) = I$

on squaring

$16a^2(1 + b)^2 = I^2$

$16(2\sqrt{m} - 1)(1 - \sqrt{m})^2 = I^2$

$16(2\sqrt{m} - 1)(1 + m - 2\sqrt{m}) = I^2$

$16\left[2m\sqrt{m} + 4\sqrt{m} - 5m - 1\right] = I^2$

$\Rightarrow$ $2m\sqrt{m} + 4\sqrt{m} - 5m - 1$

should be rational which is not possible for any integer value of m

$\Rightarrow$ b can not be irrational.

3. Show that there are infinitely many triples (x, y, z) of integers such that $x^3 + y^4 = z^{31}$.

Sol. $x^3 + y^4 = z^{11}$

Let x = 0 ⇒ $y^4 = z^{11}$ ⇒ y = $z^{\frac{11}{4}}$ (many values of z of the form $(\text{integer})^4$ will give integral values of y. so infinite sets.

4. Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Sol. $^{36}C_3 - 36 - 36\,(32) - 18 \times 30$

Total ways of 3 selections = $^{36}C_3$

Three at adjacent positions = 36

Exactly two at consecutive positions = 36 × 32

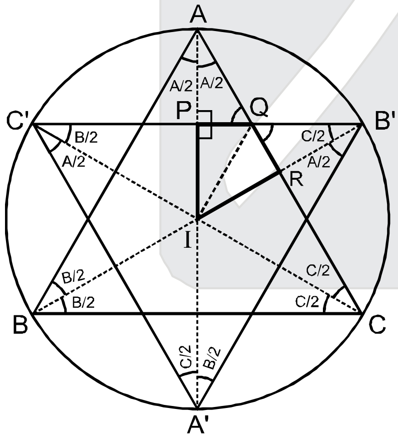
Diametrically opposite but not adjacent = $\frac{36 \times 30}{2}$

So $^{36}C_3 - 36 - 36 \times 32 - 18 \times 30$

= 5412

5. Let ABC be a triangle with circumcircle Γ and incentre I. Let the internal angle bisectors of ∠A, ∠B, and ∠C meet Γ in A′, B′ and C′ respectively. Let B′C′ intersect AA′ in P and AC in Q, and let BB′ intersect AC in R. Suppose the quadrilateral PIRQ is a kite; that is IP = IR and QP = QR. Prove that ABC is an equilateral triangle.

Sol.



I R Q P is a kite

∠ PQA = ∠ RQB (vertically opposite)

∠ QPI = ∠ QRI (kite) ⇒ ∠ APQ = ∠ QRB

⇒ Δ APQ ≅ Δ B′RQ (ASA)

⇒ $\frac{A}{2} = \frac{C}{2}$ ⇒ ∠ A = ∠ C

$\Rightarrow$ Δ ABC is isosceles, i.e. AB = BC and also Δ IAC is isosceles

so IA = IC and $\Delta A I Q \cong \Delta B' I Q$

$\Rightarrow$ $AI = B' I = C I$

i.e. I circumcentre as well incentre ΔABC is equilateral

6. Show that there are infinitely many positive real numbers a which are not integers such that a(a–3{a} is an integer. (Here {a} denotes the fractional part of a. For example {1.5} = 0.5; {–3.4} = 0.6.).

Sol. Let $a = I + f$

$\Rightarrow$ $(I + f)(I - 2f)$ should be an integer

$\Rightarrow$ $I^2 - If - 2f^2$ should be an integer

$\Rightarrow$ $I^2 - f(I + 2f)$ should be an integer

$f(I + 2f)$ should be an integer (Let = K)

$2f^2 + If - K = 0$

$\Rightarrow$ $f = \dfrac{-I \pm \sqrt{I^2 + 8K}}{4}$ (ignore – sign as 0 < f < 1)

$\Rightarrow$ $\sqrt{I^2 + 8K} - I < 4$

$\sqrt{I^2 + 8K} < 4 + I$

$I^2 + 8K < 16 + I^2 + 8I$

$8K - 8I < 16$

$K < I + 2$

For any I, we get possible values of K which makes given expression an integer.