Seat No : ________

Regional Mathematical Olympiad – 2015

Gujarat Diu, Daman & DNH Region

Time : 3 hours December 06, 2015

Instructions :

(i) On the first page of your answer – sheet write your full name (in block letters),

School name and address, your residential address (including pin code and phone numbers) and email id.

(ii) Calculators (in any form) and protectors are not allowed.

(iii) Rulers and compasses are allowed.

(iv) All questions carry equal marks. Maximum marks : 102.

(v) Answer to each question should start on a new page. Clearly indicate the question number.

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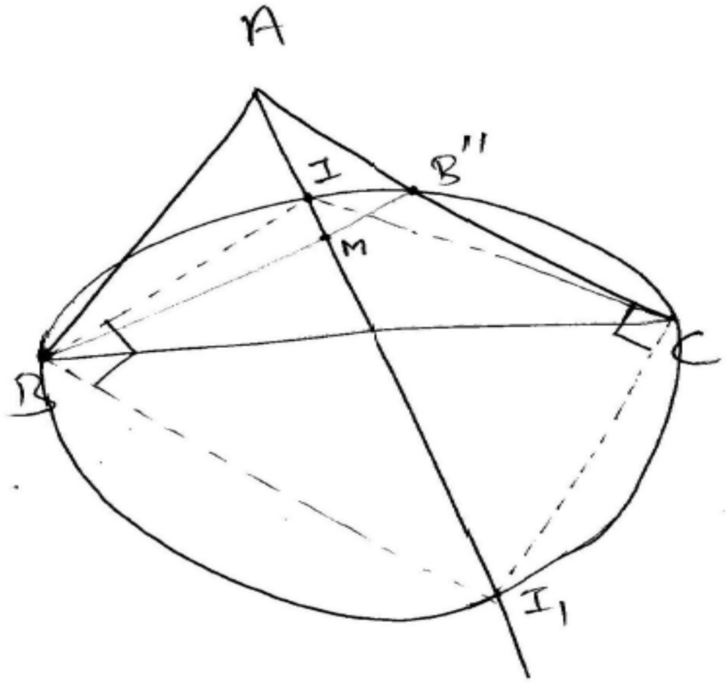
1. Let ABC be a triangle. Let B' denote the reflection of B in the internal angle bisector ℓ of $\angle A$. Show that the circumcentre of the triangle CB' I lies on the line ℓ, where I is the incentre of the triangle ABC.



Sol. In given ΔABC, Let I be the excentre and I_1 be the except corresponding to vertex A.

As we know that $CI \perp CI_1$ (external and internal angle bisectors areperpendicular)

Similarly $BI \perp BI_1$

$\angle I$ B I_1 = $\angle$ I C I_1 = 90^0

$\Rightarrow$ quadrilateral I B I_1 C is cyclic

Let us assume that circumcircle of quadrilateral I B I_1 C intersect side AC at point B" and let BB" intersect II_1 at point M

we have $\angle ICB'' = \angle IBB'' = \dfrac{C}{2}$ .... (i)

[Angle made by arc in same segment are equal and IC is the internal angle bisector of C]

Similarly $\angle ICB = \angle IBB'' = \frac{C}{2}$ ....(ii)

From (i) and (ii), we have

$\angle IBB'' = \angle IB''B = \frac{C}{2}$ .... (iii)

Also in ΔBCI_1 we have $\angle BCI_1 = \frac{\pi}{2} - \frac{c}{2}$ $\left[\because \angle ICI_1 = 90^0\right]$

Also $\angle BCI_1 = \angle BII_1 = \frac{\pi}{2} - \frac{c}{2}$ .... (iv)

Now in ΔIBM, we have

$\angle IBB'' = \angle IBM = \frac{c}{2}, \quad \angle BII_1 = \angle BIM = \frac{\pi}{2} - \frac{c}{2}$ [from (iii) & (iv)

$\Rightarrow \angle IMB = 90^0$

Similarly in $\Delta IB''M$

$\angle IB''M = \frac{c}{2}$, $\angle B''IM = \frac{\pi}{2} - \frac{c}{2}$

and $\angle IMB'' = \frac{\pi}{2}$

$\Rightarrow \Delta IBM \cong \Delta IB''M$

$\Rightarrow$ B” is the mirror image B’ of B in line II_1

Hence B” is the given reflection B’ of point B in II_1

$\Rightarrow$ I, B’ and C are concyclic with II_1 as diameter of their circum circle

i.e centre of circum circle of $\Delta IB'C$ lies on II_1 i.e bisector ℓ of $\angle A$

2. Let P(x) = $x^2 + ax + b$ be a quadratic polynomial where a is real and b $\neq 2$ is rational. Suppose $P(0)^2$, $P(1)^2$, $P(2)^2$ are integers. Prove that a and b are integers.

Sol. As b is rational number and $(P(0))^2$ is integer hence b will be an integer

Now, Let $(P(1))^2$ = $(1+a+b)^2 = I_1$ $(P(2))^2$ = $I_2 = 4+2a+b$

$I_1 = 1+a^2+b^2+2ab+2a+2b$

(1 + b^2 + 2b) + (a^2 + 2ab + 2a) = integer + I_3

$I_2 = 16+4a^2+b^2+4ab+8b+16a$

= (16 + b^2 + 8b) + (4a^2 + 4ab + 16a) = integer + I_4

$a^2+2a+2ba = I_3$

$4a^2+4ab+16a = I_4 = 2I_5$

$2I_5 - 2I_3 = 2a^2 + 12a$ = integer

$I_5 - I_3 = a^2 + 6a$

$2(a+3)^2$ = integer - 18

Hence a is an integer

3. Find the all integers a, b, c such that

$$a^2 = bc+4,\ b^2 = ca+4$$

Sol. a^2 = bc + 4 .... (I)

b^2 = 4c + 4 ....(ii)

Adding (i) and (ii)

$a^2 + b^2 = c(a + b) + 8$

subtracting(i) and (ii)

$a^2 - b^2 = c(b - a)$

(a – b) (a + b + c) = 0 ....(iii)

Case I a = b

$\Rightarrow a^2 - ac - 4 = 0$

D = $c^2 + 16 = I^2$

$I^2 - c^2 = 16$

$c = 0, \pm 3$

a	b	c
2	2	0
-2	-2	0
-1	-1	3
4	4	3
1	1	-3
-4	-4	-3

$c = 3,\quad a^2 - 3a - 4$

$a = -1, 4$

$c = -3\quad a^2 + 3a + 4 = 0$

$a = 1, -4$

Case II $a + b + c = 0$

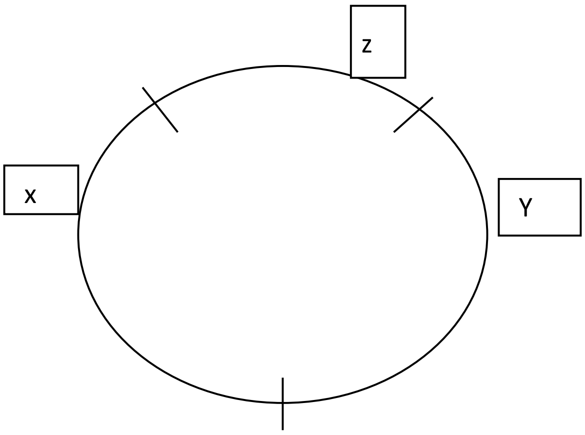
$a \neq b$

$a^2 + b^2 + c^2 = 8$ ………….. (From iii)

a	b	c
-2	2	0
2	-2	0
0	2	-2
0	-2	2
2	0	-2
-2	0	2

Total number of solution = 12

4. Suppose 40 objects are placed among a circle at equal distance. In how many ways can 3 objects be chosen from among them so that no two of the three objects are adjacent nor diametrically opposite ?



Sol. $x + y + z = 37$

$x \geq 1, y \geq 1, z \geq 1$

$\Rightarrow x = x' + 1, y = y' + 1, z = z' + 1$

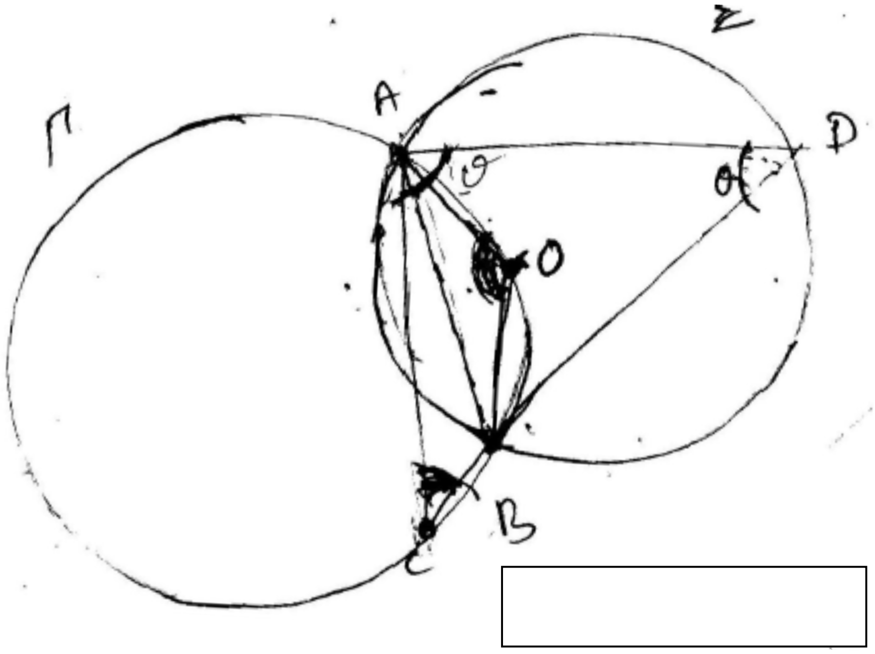
$X' + y' + z' = 34$

No of solution of ways $= \dfrac{{}^{36}C_2 - 18 \times 3}{3!} + 18 - 9 = 105$

(let a_1, a_2, a_3 ………………… a_{40} are then if x = 19 then the objects selected becomes diametrically opposite.

5. Two circles Γ and Σ intersect at two distinct points A and B. A line through B intersects Γ and Σ again at C and D respectively. Suppose that CA = CD. Show that the entre of Σ lies on Γ.

O be the centre of the circle



$\because \angle ADB = \theta$

$\therefore \angle AOB = 2\theta$

$\because AC = CD$

$\Rightarrow \angle CDA = \angle CAD = \theta$

$\therefore \angle ACD = \pi - 2\theta$

$\therefore \angle ACD + \angle AOB = \pi$

$\therefore \angle ACB + \angle AOB = \pi$

And points A, C ,B are on the circle Γ

$\therefore$ Points A, C, B, O are concyclic

Hence the centre O, of the circle $\sum$ lies on the circle Γ.

6. How many integers m satisfy both the following properties :

(i) $1 \le m \le 5000$ (ii) $\lfloor \sqrt{m} \rfloor = \lfloor \sqrt{m+125} \rfloor$?

(Here [x] denote the largest integer not exceeding x, for any real number x.)

Sol. Let $\lfloor \sqrt{m} \rfloor = p$

$p \le \sqrt{m} < p+1$

$p^2 \le m < (p+1)^2$ .... (i)

$\Rightarrow p^2 \le m+125 < (p+1)^2$ .... (ii)

$\Rightarrow p^2 \le m$

$(p+1)^2 > m+125$

$m < p^2 + 2p - 124$

$p^2 + 2p - 124 < 5000$

$(p+1)^2 < 5125$

1 + P < 71.59

P < 70.59

$\lfloor \sqrt{m} \rfloor = p$

$p^2 < m < (p+1)^2$

$p^2 < m+125 < (p+1)^2$

$m < p^2 + 2p - 124$

2p + 1 > 125

P > 62

Hence total value of m

= $\sum_{63}^{70} (2p-124)$ = 72

Note :

(i) For any query for Mathematics Olympiad in Gujarat, Contact Coordinator, Regional Mathematical Olympiad after 31st December through email:gaint_spardha@yahoo.co.in

(ii) For the result of RMO – 2015 visit the website https:/sites.google.com/site/rmogujarat.

(iii) If you Select for INMO – 2015 you will be informed by email / phone.