Regional Mathematical Olympiad-2015
क्षेत्रीय गणित ओलिंपियाड-2015

Time : 3 hours (समय: 3 घंटा) **December 06, 2015 (दिसम्बर 06, 2015)**

Instructions (निर्देश) :

- **Calculators (in any form) and protractors are not allowed.**
  किसी भी तरह के गुणक **(Calculators)** तथा चांदा के प्रयोग की अनुमति नहीं है।
- **Rulers and compasses are allowed.**
  पैमाना **(Rulers)** तथा परकार **(compasses)** के प्रयोग की अनुमति है।
- **Answer all the questions. All questions carry equal marks. Maximum marks : 102**
  सभी प्रश्नों के उत्तर दीजिये। सभी प्रश्नों के अंक समान हैं, अधिकतम् अंक : **102**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
  प्रत्येक प्रश्न का उत्तर नए पेज से प्रारंभ कीजिये। प्रश्न क्रमांक स्पष्ट रूप से इंगित कीजिये।

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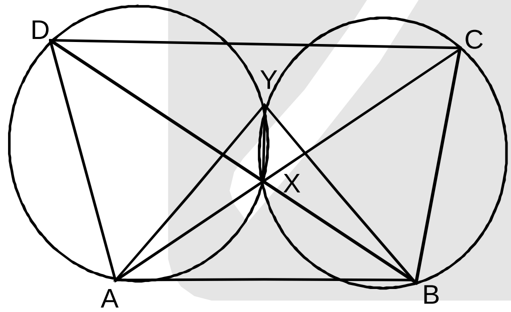
1. In a cyclic quadrilateral ABCD, let the diagonals AC and BD intersect at X. Let the circumcircles of triangles AXD and BXC intersect again at Y. If X is the incentre of triangle ABY, show that $\angle CAD = 90°$

Sol.



ABCD is cyclic quadrilateral and X is incentre of ΔABY

Let

$\angle YAX = \angle BAX = \alpha$

$\angle YBX = \angle ABX = \beta$

$\angle AYX = \angle BYX = \gamma$

So $\angle DBA = \angle DCA = \beta$ (Angle is the same segment)

$\angle BAC = \angle BDC = \alpha$ (Angle is the same segment)

$\angle AYX = \angle ADX = \gamma$ (Angle is same segment)

Now is ΔABY

$2\alpha + 2\beta + 2\gamma = 180°$

$\alpha + \beta + \gamma = 90°$ ...(1)

In ΔCAD

$\angle CAD + \angle ADC + \angle DCA = 180°$

$\angle CAD + \alpha + \gamma + \beta = 180°$

$\angle CAD = 90°$

2. Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n)$, $P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

Sol. $P_1(x) = x^2 + a_1x + b_1$

$P_2(x) = x^2 + a_2x + b_2$ $\quad$ a_1, b_1, a_2, b_2 are integers

$a_1 \neq a_2$

$m \neq n$

$P_1(m) = P_2(n)$

$m^2 + a_1m + b_1 = n^2 + a_2n + b_2$

$(m^2 - n^2) + a_1m - a_2n + b_1 - b_2 = 0$ $\quad$ ..(1)

$P_2(m) = P_1(n)$

$n^2 + a_1n + b_1 = m^2 + a_2m + b_2$

$(m^2 - n^2) + a_2m - a_1n + b_2 - b_1 = 0$ $\quad$ ..(2)

from (1) & (2)

$b_2 - b_1 = (m^2 - n^2) + a_1m - a_2n = -(m^2 - n^2) - a_2m + a_1n$

$2(m^2 - n^2) + m(a_1 + a_2) - n(a_1 + a_2) = 0$

$2(m^2 - n^2) + (a_1 + a_2)(m - n) = 0$

$(m - n)[2(m + a) + a_1 + a_2] = 0$

$m - n \neq 0$ hence $[2(m + n) + a_1 + a_2] = 0$

$\therefore 2(m + n) + a_1 + a_2 = 0$

$(m + n) = -\dfrac{a_1 + a_2}{2}$

Now, as m, n are integers so (m + n) is also an integer and $\dfrac{a_1 + a_2}{2} \in I$

$\therefore a_1 + a_2$ must be even integer. It is possible only when both a_1 and a_2 are even or odd. In both the cases we get $(a_1 - a_2)$ always be even.

3. Find all fractions which can be written simultaneously in the forms $\dfrac{7k-5}{5k-3}$ and $\dfrac{6l-1}{4l-3}$, for some integers k,l.

Sol. $\dfrac{7k-5}{5k-3} = \dfrac{6\ell-1}{4\ell-3}$

$28k\ell - 21k - 20\ell + 15 = 30k\ell - 5k - 18\ell + 3$

$2k\ell + 16k + 2\ell - 12 = 0$

$k\ell + 8k + \ell - 6 = 0$

$k\ell + 8k + \ell = 6$

$k(\ell + 8) + \ell + 8 = 14$

$(k + 1)(\ell + 8) = 14$

$(k + 1)(\ell + 8) = 14 = 14 \times 1 = 7 \times 2 = -14 \times -1 = -7 \times -2$

if $k + 1 = 14$ and $\ell + 8 = 1$ or $k + 1 = 1$ and $\ell + 8 = 14$

$(k = 13, \ell = -7)$ $\quad$ $(k = 0, \ell = 6)$

in the same way we can find the other solution

$(k, \ell) = (13, -7), (-15, -9), (0, 6), (-2, -22), (6, -6), (-8, -10), (1, -1), (-3, -15)$

so total 8 solutions

Ans. $\dfrac{43}{31}, \dfrac{55}{39}, \dfrac{5}{3}, \dfrac{19}{13}, \dfrac{37}{27}, \dfrac{61}{43}, 1, \dfrac{13}{9}$

4. Suppose 28 objects are placed along a circle at equal distances, In how many ways can 3 objects be chosen from among them no two of the three chosen objects are adjacent not diametrically opposite ?

Sol. 1st point can be selected in 28 ways.

Total number of ways of selecting three point from which no two are adjacent = $\frac{^{28}C_1(^{25}C_2 - 24)}{3}$ =2576

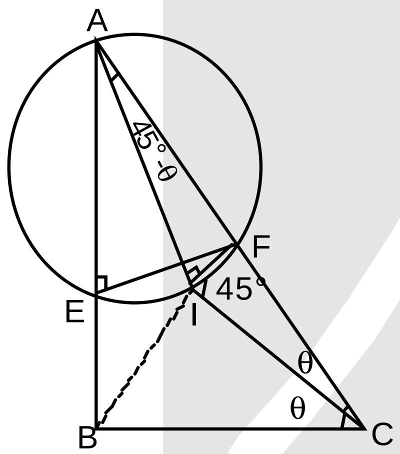
Number of ways in which points are diametrically opposite = 14 × 22 = 308

Required number of ways = 2576 – 308 = 2268

5. Let ABC be a right triangle with $\angle B = 90°$. Let E and F be respectively the mid-points of AB and AC. Suppose the incentre I of triangle ABC lies on the circumcircle of triangle AEF. Find the ratio BC/AB.

Ans. $\frac{4}{3}$

Sol.



$\angle AEF = \angle ABC = 90°$ <EF || BC> and $EF = \frac{BC}{2}$

So AF is diameter of the circumcircle of ΔAEF

$\Rightarrow \angle AIF = \angle AEF = 90°$

$\angle CIF = \angle AIC - \angle AIF$

$= 135° - 90° = 45°$ ($\angle AIC = 135$ as I is the incentre of ΔABC)

Now, $AF = FC = \frac{1}{2}AC$ (F is the mid point of AC)

Let $\angle FCI = \angle BCI = \theta$

So, $\angle IAC = 180 - (\angle AIC + \angle ICA)$

$= 180 - (135 + \theta)$

$= 45 - \theta$

Apply sin rule is ΔCIF

$\frac{\sin\theta}{IF} = \frac{\sin 45°}{CF} \Rightarrow IF = CF \sin\theta \sqrt{2}$ ..(1)

In ΔAIF

$\frac{\sin(45-\theta)}{IF} = \frac{\sin 90°}{AF} \Rightarrow IF = \sin(45-\theta)\, AF$ ..(2)

From Equation (1) and (2)

$\sqrt{2}\, CF \sin\theta = \sin(45 - \theta)\, AF$

$\sqrt{2} = \dfrac{\sin(45-\theta)}{\sin\theta}$

$\sqrt{2} = \dfrac{\cos\theta - \sin\theta}{\sqrt{2}\sin\theta} \Rightarrow \tan\theta = \dfrac{1}{3}$

Now, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \dfrac{2}{3}\times\dfrac{9}{8} = \dfrac{3}{4}$

In ΔABC $\tan\angle ABC = \tan 2\theta = \dfrac{3}{4} = \dfrac{AB}{BC} \Rightarrow \dfrac{BC}{AB} = \dfrac{4}{3}$

6. Find all real numbers a such that 3 < a < 4 and a(a – 3{a}) is an integer (Here {a} denotes the fractional part of a. For example {1.5} = 0.5 ; {–3.4} = 0.6).

Sol. $3 < a < 4$

Let $a = 3 + f$

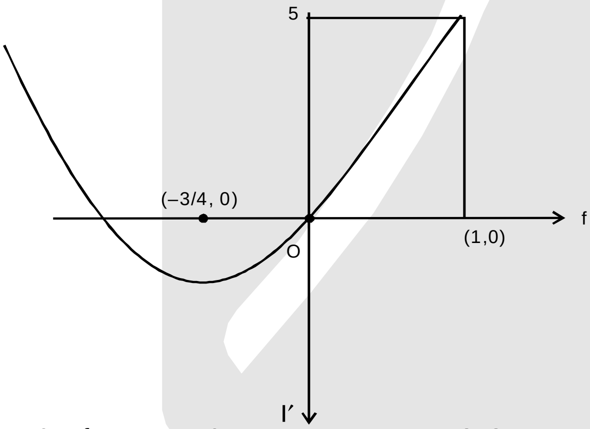
$a(a - 3\{a\}) = I$

$(3 + f)(3 + f - 3f) = I$

$(3 + f)(3 - 2f) = I$

$9 - 3f - 2f^2 = I$

$2f^2 + 3f = I'$



as $0 < f < 1$ $\therefore$ $0 < I' < 5 \Rightarrow I' = 1, 2, 3, 4$

$3f + 2f^2 = 1, 2, 3, 4.$

Case-1 : $2f^2 + 3f = 1$

$f = \dfrac{-3+\sqrt{17}}{4} \Rightarrow a = 3 + f = \dfrac{9+\sqrt{17}}{4}$

in the same way we can find other value of a

$a = \dfrac{9+\sqrt{17}}{4}, \dfrac{7}{2}, \dfrac{9+\sqrt{33}}{4}, \dfrac{9+\sqrt{41}}{4}$

Total 4 solutions.