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PRMO
2019

NATIONAL BOARD FOR HIGHER MATHEMATICS
AND
HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

MATHEMATICS TEACHERS' ASSOCIATION
MTA Pre-REGIONAL MATHEMATICAL OLYMPIAD (PRMO), 2019

QUESTION PAPER WITH SOLUTION & ANSWER KEY

Date: 25th August, 2019 | Time: 10:00 AM to 1:00 PM

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SUCCESS AT JEE ADVANCED 2019

TOTAL SELECTIONS

5162

1 or 2 Yearlong Classroom: **3473** | Distance Learning & e-Learning: **1689**
Kota Classroom : **2245** | All Study Centres (Classroom): **1228**

AIR 73
Ananjan Nandi
Classroom Student since Class XI

AIR 80
Tamajit Banerjee
Classroom Student since Class XII

List of all our selected students is available on our official website

HIGHEST* CLASSROOM GIRL STUDENTS SELECTED	HIGHEST* CLASSROOM HINDI MEDIUM STUDENTS SELECTED
353	**277**

AIR 127

SAHASRA RANJAN
Classroom Student since class XII

AIR 145

ANUBHAV KALYANI
Classroom Student since class XI

AIR 161

ANGIKAR GHOSAL
Classroom Student since class XII

AIR 168

KRITIN SHARMA
Classroom Student since class XI

AIR 179

ATREYA GOSWAMI
Classroom Student since class XI

AIR 192

SAPTARSHI DASGUPTA
Classroom Student since class XI

AIR 212

ATUR GUPTA
Classroom Student since class VIII

AIR 216

SHUBHANKAR
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AIR 225

AMAN GUPTA
Classroom Student since class IX

AIR 237

KANISHK SINGHAL
Classroom Student since class VIII

AIR 247

RUPINDER GOYAL
Classroom Student since class XI

BEST RANKS IN CATEGORIES

AIR 3

SC
ANSHUL NAVPHULE
Classroom Student since class VII

AIR 4

ST
ATIN BAINADA
Classroom Student since class XI

AIR 11

OBC NCL
SAHASRA RANJAN
Classroom Student since class XII

AIR 21

GEN EWS
SOUMIL SARAWGI
Classroom Student since class XI

*Based on the information collected from public domain till 17th June, 2019, 1:00 PM

Top 100 AIRs - Other Categories from Classroom Programs

Gen - EWS	21, 22, 23, 37, 42, 43, 54, 90, 94
OBC - NCL	11, 34, 40, 56, 72, 73, 76
SC	3, 11, 30, 31, 36, 37, 53, 64, 72, 92, 94, 100
ST	4, 10, 13, 18, 21, 22, 30, 37, 43, 53, 68, 70, 74, 83, 89, 90, 91, 94

Top 100 AIRs
Distance Learning Program

18	42	48	54
58	61	90	98

All from General Category

JNV Bundi Result Highlight

HIGHEST* SELECTION RATIO
amongst any JNV across India

84% 84 students selected out of 100 students appeared

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INSTRUCTION

Number of Questions: 30 **Max. Marks: 102**

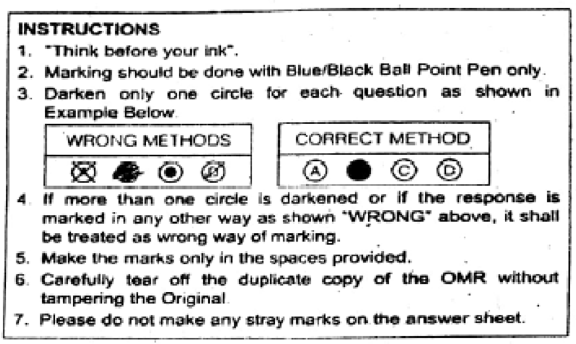
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machine through scanning. On OMR sheet, darken bubbles completely with a black pencil or a black blue pen. Darken the bubbles completely only after you are sure of your answer: else, erasing lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address and date of birth entered on the OMR sheet will be your login credentials for accessing your PROM score.

4. Incompletely, incorrectly and carelessly filled information may disqualify your candidature.

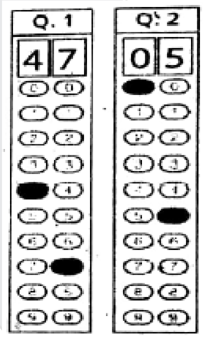
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below

WRONG METHODS	CORRECT METHOD
	Ⓐ ● Ⓒ Ⓓ

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darken bubble will be considered as your final answer.

7. Questions 1 to 6 carry 2 marks each: Questions 7 to 21 carry 3 marks each: Questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. Consider the sequence of number $\left[n + \sqrt{2n} + \frac{1}{2}\right]$ for $n \geq 1$, where [x] denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \ldots$ then find the n_{12}.

Sol. **(78)**

$S_n = n + \left[\sqrt{2n} + 0.5\right]$, $n \geq 1$

[·] = G.I.F.

n	$\sqrt{2n}$	$\left[\sqrt{2n}+0.5\right]$	$n+\left[\sqrt{2n}+0.5\right]$
1	1.4	1	2
2	2	2	4
3	2.4	2	5
4	2.8	3	7
5	3.1	3	8
6	3.4	3	9
7	3.7	4	11
8	4	4	12
9	4.2	4	13
10	4.4	4	14
11	4.6	5	16
12	4.8	5	17
13	5.0	5	18
14	5.2	5	19
15	5.4	5	20
16	5.6	6	22
17	5.8	6	23
18	6	6	24
19	6.1	6	25
20	6.3	6	26
21	6.4	6	27
22	6.6	7	29

By observing pattern,

Missing numbers = $1, \; 3, \; 6, \; 10, \; 15, \; 21, \; 28, \; 36, \; 45, \; 55, \; 66, \; 78, \ldots$

$+2 \quad +3 \quad +4 \quad +5 \quad +6$

$\therefore$ 12th number in series = 78

2. If $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

Sol. **(93)**

$x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ root of equation :

$x^4 + ax^3 + bx^2 + cx + d = 0$; a, b, c, d $\in$ Z

Now,

$$\left(x - \sqrt{2}\right)^2 = \left(\sqrt{3} + \sqrt{6}\right)^2 \qquad \Rightarrow \qquad x^2 - 2\sqrt{2}x + 2 = 9 + 6\sqrt{2}$$

$$\Rightarrow \qquad x^2 - 7 = 2\sqrt{2}\,(3 + x)$$

(On squaring both sides)

$\Rightarrow \quad x^4 - 14x^2 + 49 = 8\left(x^2 + 6x + 9\right)$

$\Rightarrow \quad x^4 - 22x^2 - 48x - 23 = 0$

$\therefore$ a = 0, b = – 22, c = – 48 and d = – 23

$\therefore$ |a + b + c + d| = 93

3. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Sol. **(25)**

Let, c $\rightarrow$ be positive integer $\le$ 100 such that $c^2 = a^2 - b^2$; a, b $\in$ Z.

C – I : Difference of a and b is 1.

$\therefore \quad 1^2 - 0^2 = 1$

$2^2 - 1^2 = 3$

$3^2 - 2^2 = 5$

$4^2 - 3^2 = 7$

$5^2 - 4^2 = 9$

$\vdots$

$50^2 - 49^2 = 99$

All odd number upto 100 can be expressed as difference of two squares.

C – II : Difference of a, b is 2

$2^2 - 0^2 = 4$

$3^2 - 1^2 = 8$

$4^2 - 2^2 = 12$

$\vdots$

$26^2 - 24^2 = 100$

All multiples of '4' upto 100 can be expressed as difference of two squares.

C – III : Difference of a, b is 3

This case will give odd numbers which are already counted

C – IV : Difference of a, b is 4

This case will give multiples of 4 which are already counted

Similarly, the remaining cases will all give number already counted in

C – I and C – II

$\therefore$ No. of numbers which can be expressed as difference of squares

= (odd numbers) + (multiples of 4) = 50 + 25 = 75

$\therefore$ Required numbers which cannot be expressed as a difference of squares of two integers are = 100 – 75 = 25

4. Let $a_1 = 24$ and form the sequence a_n, $n \geq 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

24, 2534, 253534, 25353534, ……..

What is the least value of n for which a_n is divisible by 99?

Sol. **(88)**

$a_1 = 24$ & $a_n = 100 . a_{n-1} + 134$

First few terms :

$a_1 = 24$

$a_2 = 2534$

$a_3 = 253534$

$a_4 = 25353534$

$\therefore a_n = 2\underbrace{535353.......53}_{(n-1)\text{Times'53'}}4$

Now, an $\rightarrow$ divisible by 99 $\Rightarrow$ by 9 & 11 both

Sum of digits = 6 + 8 (n – 1)

To be divisible by 9,

n = 7, 16, 25, 34, 43, 52, 61, 70, 79, 88, ….

$a_7 = 2\underbrace{535353535353}_{6\text{Times'53'}}4$

But $a_7 \rightarrow$ Not divisible by 11.

$a_{16} = 2\underbrace{5353535353......53}_{15\text{Times'53'}}4$

Similarly, $a_{16} \rightarrow$ Not divisible by 11.

Now, n = 88

$a_{88} = 2\underbrace{5353......53}_{87\text{Times'53'}}4$

Divisibility by $11 \rightarrow |(2+3+3.....)-(5+5+....4)|$

$= |263 - 439|$

$= 176$

$\therefore$ Least n = 88

5. Let N be the smallest positive integer such that $N + 2N + 3N + ... + 9N$ is a number all whose digits are equal. What is the sum of the digits of N?

Sol. **(37)**

$N \in Z^+$

P $=$ N + 2N + 3N + ……… + 9N = 45N

$45 = \underbrace{555............5}_{a\text{ times}}$

Only 55 ………. 5 can be repeated as 45N will have units place = 0 or 5

Also as 45N $\rightarrow$ multiple of '9' also.

$\therefore$ If a = 9 $\Rightarrow \frac{555555555}{45} = N$

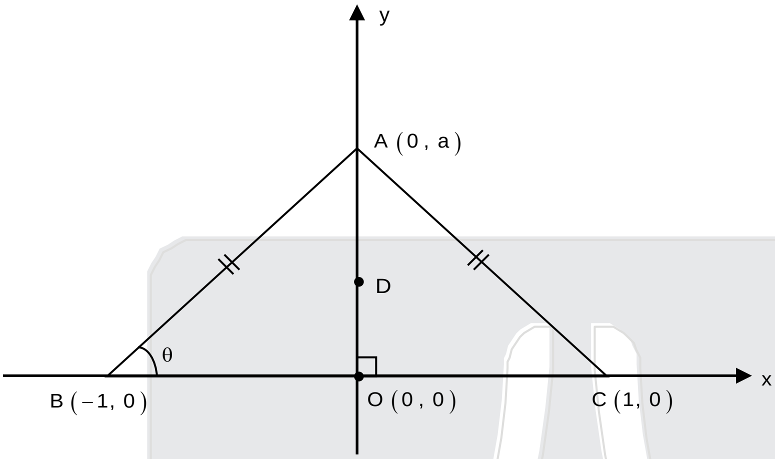
$\therefore$ N $=$ 12345679

Sum of digits of N = 37.

6. Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of ABC at B is perpendicular of AC. Find $\angle ABC$ measured in degrees.

Sol. **(30)**

Consider the Isosceles triangle with vertex A, B, C such that AB = AC



By symmetry $\rightarrow$ circumcentre will lie on y – axis

$\therefore$ D $\equiv$ (0 , k) $\rightarrow$ circumcenter

Radius of circumcircle = DA = DB = DC

$\therefore \sqrt{(a-k)^2} = \sqrt{k^2+1}$

$\Rightarrow a^2 + k^2 - 2ak = k^2 + 1$

$\Rightarrow a^2 - 2ak = 1$ ...(1)

Now

Circumcircle : $x^2 + (y-k)^2 = k^2 + 1$

$\therefore$ Slope of tangent at B,

$2x + 2(y-k)y' = 0$

$\Rightarrow y'\big|_B = \left.\dfrac{-x}{y-k}\right|_B = \dfrac{1}{-k}$

Also, $m_{AC} = -a$

By given condition, $(-a) \times \left(-\dfrac{1}{k}\right) = -1$

$\Rightarrow a = -k$ ...(2)

From (1) & (2), $a^2 + 2a^2 = 1$

$\Rightarrow a = \pm \dfrac{1}{\sqrt{3}}$

If $a = \dfrac{1}{\sqrt{3}}$ $\qquad \Rightarrow k = -\dfrac{1}{\sqrt{3}}$

$\therefore \tan\theta = m_{AB} = \dfrac{1}{\sqrt{3}}$ $\qquad \Rightarrow \theta = 30°$

$\Rightarrow \angle ABC = 30°$

7. Let s(n) denote the sum of the digits of a positive integer n in base 10. If s(m) = 20 and s(33m) = 120, what is the value of s(3m)?

Sol. **(60)**

S(m) = 20 and S (33m) = 120 is possible only for/m/ having digits ‘0’ or ‘1’

$\therefore$ S(3m) = 60

8. Let $F_k(a,b) = (a+b)^k - a^k - b^k$ and let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. For how many ordered pairs (a, b) with a, b $\in$ S and $a \le b$ is $\frac{F_5(a,b)}{F_3(a,b)}$ an integer?

Sol. **(22)**

$$\frac{F_5(a,b)}{F_3(a,b)} = \frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$$

$$= \frac{5}{2}\left(a^2 + b^2 + ab\right) \qquad \in I$$

$a = 3k_1$ or $3k_1+1$ or $3k_2 + 2$

and $b = 3k_2$ or $3k_2+1$ or $3k_2 + 2$

only when a and b give same remainder $a^2 + b^2 + ab$ is divisible by 3.

$\therefore$ a = 1, b = 1, 4, 7, 10 $\rightarrow$ 4

a = 2, b = 2, 5, 8 $\rightarrow$ 3

a = 3, b = 3, 6, 9 $\rightarrow$ 3

a = 4, b = 4, 7, 10 $\rightarrow$ 3

a = 5, b = 5, 8 $\rightarrow$ 2

a = 6, b = 6, 9 $\rightarrow$ 2

a = 7, b = 7, 10 $\rightarrow$ 2

a = 8, b = 8 $\rightarrow$ 1

a = 9, b = 9 $\rightarrow$ 1

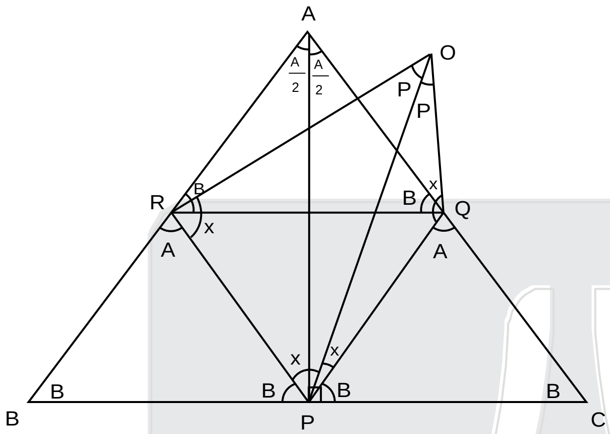
a = 10, b = 10 $\rightarrow$ 1

Total 22 ordered pair

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α° and the smaller one β° then what is the value of $\alpha - \beta$?

Sol. **(90)**

Sol.



A + 2B = 180

$\frac{A}{2} = 90 - B$

2x + 28 = 180

x = 90 – B

$x = \frac{A}{2}$

$\Rightarrow OP \perp BC$

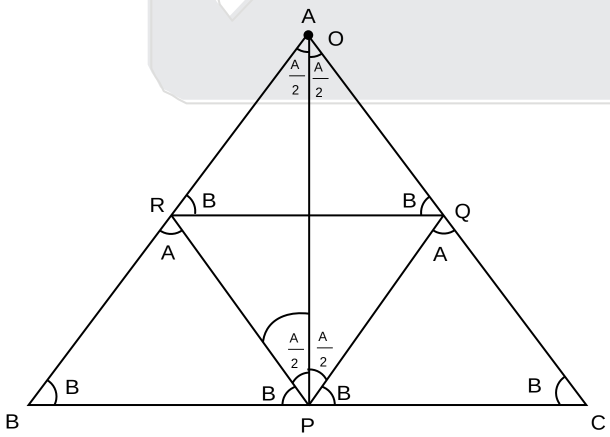
$\therefore$ O and A coincide

$\therefore \angle ORP = \angle OPR = \angle RAP = 60^\circ$

$\therefore \alpha = 120^\circ \; \beta = 30^\circ$

$\alpha - \beta = 90^\circ$

By drawing properly we get



10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

Sol. **(42)**

At x = ab minutes past 5 hr = 5 x 60 + 10 a + b minutes

At y = bc minutes past 6 hr = 6 x 60 + 10 b + a minutes

Total minutes of walk = 60 + 9b – 9a = 10 a+b

= 60 + 8b = 19a

a = 4, b = 2

I walked for 42 minutes

11. Find the largest value of a^b such that the positive integers a, b > 1 satisfy $a^b b^a + a^b + b^a = 5329$.

Sol. **(81)**

$(a^b + 1)(b^a + 1) = 5330 = 2 \times 5 \times 13 \times 41$

$\Rightarrow \quad a^b = 81 = 3^4$ and $b^a = 64 = 4^3$

Or $a^b = 64 = 4^3$ and $a^b = 81 = 3^4$

$\therefore a^b = 81$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$\{10a + b : 1 \le a \le 5, 1 \le b \le 5\}$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Sol. **(47)**

10a + b; $1 \le a \le 5$, $1 \le b \le 5$

Let us divide numbers into different sets, such as

Set 1 = {11, 12, 13, 14, 15}

Set 2 = {21, 22, 23, 24, 25}

Set 3 = {31, 32, 33, 34, 35}

Set 4 = {41, 42, 43, 44, 45}

Set 5 = {51, 52, 53, 54, 55}

Now to make a number having no two digits and no two ten's digit same, we can select any 1 number from each of set 1, set 2, set 3, set 4, set 5.

$\therefore$ No. of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1 = 120$

$\therefore \quad 120 \div 73$

$\Rightarrow$ Remainder = 47

13. Consider the sequence

1, 7, 8, 49, 50, 56, 57, 343, …

which consists of sums of distinct powers of 7, that is, 7^0, 7^1, 7^0+7^1, 7^2, …, in increasing order. At what position will 16856 occur in this sequence?

Sol. **(36)**

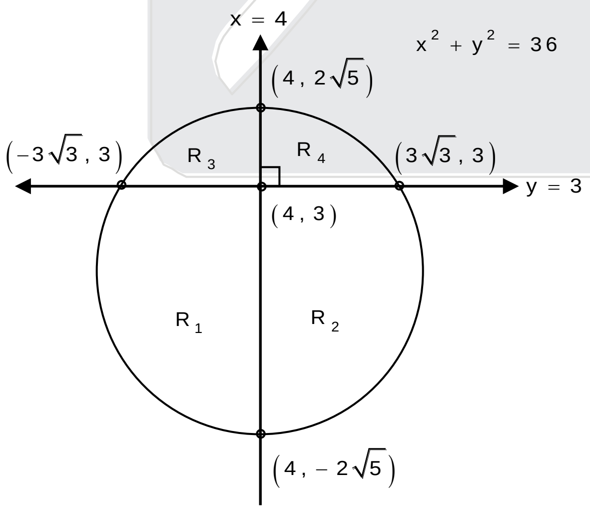
7^0	7^0+7^3	7^2+7^4	7^5
7^1	7^1+7^3	7^3+7^4	7^0+7^5
7^0+7^1	7^2+7^3	$7^0+7^1+7^4$	7^1+7^5
7^2	$7^0+7^1+7^3$	$7^0+7^2+7^4$	7^2+7^5
7^0+7^2	$7^0+7^2+7^3$	$7^0+7^3+7^4$	$=16856$
7^1+7^2	$7^1+7^2+7^3$	$7^1+7^2+7^4$	
$7^0+7^1+7^2$	$7^0+7^1+7^2+7^3$	$7^1+7^3+7^4$	
7^3	7^4	$7^0+7^1+7^2+7^4$	
	7^0+7^4	$7^0+7^1+7^3+7^4$	
	7^1+7^4	$7^0+7^2+7^3+7^4$	
		$7^1+7^2+7^3+7^4$	
		$7^2+7^3+7^4$	

i.e. we get 16856 at 36th position

14. Let R denote the circular region in the xy-plane bounded by the circle $x^2+y^2=36$. The lines x = 4 and y = 3 divide R into four regions R_i, i = 1, 2, 3, 4. If $|R_i|$ denotes the are of the region R_i and if $|R_1|>|R_2|>|R_3|>|R_4|$, determine $|R_1|-|R_2|-|R_3|+|R_4|$.

[Here $|\Omega|$ denotes the area of the region Ω in the plane.]

Sol. **(48)**



$$R_1=\int_{-2\sqrt{5}}^{3}\sqrt{36-y^2}\,dy=\frac{1}{2}y\sqrt{36-y^2}+\frac{1}{2}(36)\sin^{-1}\frac{y}{6}$$

Similarly for using integration. we can find

$R_1 + R_4 = \frac{1}{2}\pi(6)^2 + 2(3)(4)$

$\therefore$ Remaining circle area = $R_2 + R_3 = \frac{1}{2}\pi(6)^2 - 2(3)(4)$

Hence,

$R_1 + R_4 - (R_2 + R_3) = 24 + 24 = 48$

15. In base – 2 notation, digits are 0 and 1 only and the places go up in powers of – 2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base – 2 how many non zero digits does ii contain?

Sol. **(06)**

Since 2019 is closest to $2^{11} = 2048$

But since base is -2

Hence we will require $(-2)^{12}$ as maximum no. as well as $(-2)^{11}$ to compensate

The series can then be worked out as the following permutation.

1100000100111

Hence no. of non zero digits = 6

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

Sol. **(69)**

Since $P > \sqrt{n}$ and $P > 20$

Hence for the times from 2 to 20 we calculate possibilities that that $n < P^2$

Prime No.	Natural Number n
2	2
3	3, 6
5	5, 10, 15, 20
7	7, 14, 21, 28, 35, 42
11	11, 22, 33, 44, 55, 66, 77, 88, 99, 110
13	13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156
17	17, 34, 51, 68, 85, 102, 119, 136, 153, 170, 187, 204, 221, 238, 255, 272
19	19, 38, 57, 76, 95, 114, 133, 152, 171, 190, 209, 228, 247, 266, 285, 304, 323, 342

Total numbers = 69

17. Let a, b, c be distinct positive integers such that b + c – a, c + a – b and a + b – c are all perfect squares. What is the largest possible value of a + b + c smaller than 100?

Sol. **(91)**

b + c – a, c + a – b, a + b – c …... integer

Add these three

b + c – a + c+ a – b + a+ b+ c= a + b+ c

$\because$ a + b + c < 100

$\because$ Possible value of b + c – a , c + a – b, a + b – c are 1, 4, 9, 16, 25, 36, 49, 64, 81.

Either b + c – a, a + b – c, c + a – b all three will odd or all will even.

$\therefore$ Largest possible value of a + b + c is possible.

If b + c – a = 81

c + a – b = 9

a + b – c = 1

$\therefore$ a + b + c = 91

18. What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?

Sol. **(43)**

$p^3 + 4p^2 + 4p$

$= p(p^2 + 4p + 4)$

$= p\,'(p+2)^2$

Number of divisors for $a^m.b^n.c^p$…. is (m + 1) (n + 1) (p+1) …

For 30 divisors we will need 15 divisors from $(p+2)^2$

If p = 43 [for less than 43 we will have maximum 5 divisors] number of divisors

$(p+2)^2 = (43+2)^2 - 45^2$

$= 3^{2\times 2} \times 5^2$

$= 3^2 \times 2^2$

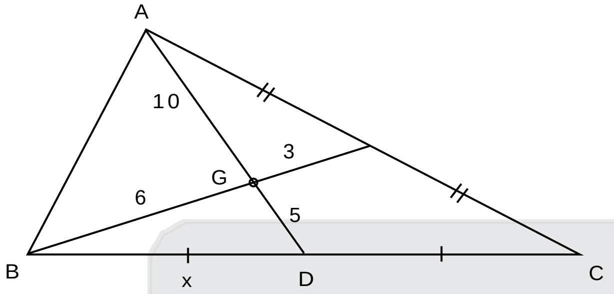
$\therefore$ Number of divisors = (4 + 1) (2 +1) = 15

$\therefore$ Ans. is 43

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

Sol. **(90)**

Area of ΔBDG



$S = \dfrac{11+x}{2}$

Area $= \sqrt{\dfrac{11+x}{2}\left(\dfrac{11+x}{2}-x\right)\left(\dfrac{11+x}{2}-5\right)\left(\dfrac{11+x}{2}-6\right)}$

$= \sqrt{\dfrac{11x}{2}\left(\dfrac{11-x}{2}\right)\left(\dfrac{x+1}{2}\right)\left(\dfrac{x-1}{2}\right)}$

$= \sqrt{\dfrac{(121-x^2)(x^2-1)}{16}}$

Let $y = \dfrac{(121-x^2)(x^2-1)}{16}$

For max area $\dfrac{dy}{dx} = 0$

$-2x(x^2-1)+(121-x^2)2x = 0$

$-x^2+1+121-x^2 = 0$

$2x^2 = 122$

$x^2 = 61$

Area of $\Delta BDG = \sqrt{\dfrac{(121-61)(61-1)}{4}}$

$= \sqrt{\dfrac{60\times 60}{4}} = \dfrac{60}{4} = 15$

Area of ΔABC = 6 x 15 = 90

20. How many 4-digit numbers $\overline{abcd}$ are there such that a < b < c < d and b – a < c – b < d – c?

Sol. **(07)**

a < b < c < d

b – a < c – b < d – c

possibilities are

a = 1, b = 2, c = 4, d = not possible

a = 1, b = 2, c = 4, d = 7

a = 1, b = 2, c = 4, d = 8

a = 1, b = 2, c = 4, d = 9

a = 1, b = 2, c = 5, d = 9

a = 1, b = 3, c = 6, d = no possible

a = 2, b = 3, c = 5 d = 8
a = 2, b = 3, c = 5 d = 9

a = 2, b = 3, c = 6, d = not possible

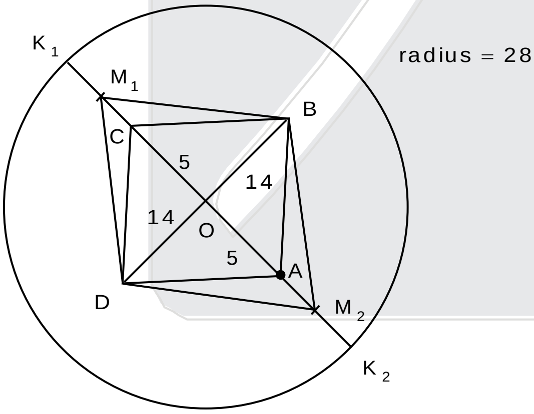
a = 2, b = 3, c = 6, d = 9
a = 4, b = 5, c = 7, d = not possible
Ans. 7

21. Consider the set E of all positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

Sol. **(Bonus)**
Remainder will be less than 11, only possible set of remainders is 1,2,4 or 2,4,8 or 1,3,9. But r greater than 1 so 2, 4, 8. But Since E is set of numbers, how can we find the sum of digits of E.
They meant "sum of digits of N.
But question should have been
"Smallest element of E is N" and sum of N.
Then there is answer. 74 : for ratio 2,4,8 as ratio greater than 1
So Bonus

22. In parallelogram ABCD, AC = 10 and BD = 28. The points K and L in the plane of ABCD move in such a way that AK = BD and BL = AC. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Sol. **(02)**

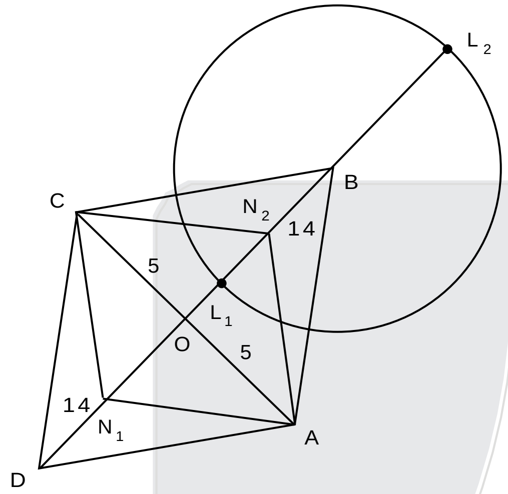


$CK_1 = 18$
$OM_1 = 5 + 9 = 14$
$\therefore \quad \angle BM_1D = 90°$ ($\angle$ in semicircle)
Similarly, $CK_2 = 10 + 28 = 38$
$\therefore \quad CM_2 = 19$
$AM_2 = 9$
$OM_2 = 5 + 9 = 14$
$\therefore \quad \angle BM_2D = 90°$ ($\angle$ in semicircle)
If K_1 or K_2 moves $\angle$ BMD will increase

By observation : $\frac{\angle BMD}{2}\uparrow$ $\quad\therefore\ \cot\left(\frac{\angle BMD}{2}\right)\downarrow$

$\therefore$ For maximum $\angle$ BMD = 90°

$$\cot^2\left(\frac{\angle BMD}{2}\right) = 1$$



$BL_1 = 10$, $OL_1 = 4$, $DL_1 = 18$, $DN_1 = 9$, $ON_1 = 5$

$\therefore\ \angle AN_1C = \frac{\pi}{2}$ ($\angle$ in semicircle)

For L_2 point,

$DL_2 = 14 + 14 + 10$

$DL_2 = 38$

$DN_2 = 19$

$ON_2 = 5$

$\angle AN_2C = \frac{\pi}{2}$ ($\angle$ in semicircle)

Now if L_1 or L_2 moves,

$\angle ANC \downarrow$ $\quad \frac{\angle ANC}{2}\downarrow$ $\quad \tan\frac{\angle ANC}{2}\downarrow$ (Observation)

Hence for maximum,

$$\angle ANC = \frac{\pi}{2}$$

$$\therefore\quad \tan^2\left(\frac{\pi}{4}\right) = 1$$

$\therefore$ Maximum = 1 + 1 = 2

23. Let t be the area of a regular pentagon with each side equal to 1. Let P(x) = 0 be the polynomial equation with least degree, having integer coefficients, satisfied by x= t and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{M}$? ($\sin 18° = \left(\sqrt{5}-1\right)/2$).

Sol. **(16)**

Area of regular pentagon $= \dfrac{a^2 n}{4\tan\left(\dfrac{180}{n}\right)}$

n $\Rightarrow$ No. of sides

a $\Rightarrow$ Length of side

$\therefore$ For regular pentagon by side length 1,

Area (t) $= \dfrac{5}{4\tan 36°}$

$= \dfrac{5}{4(0.73)}$

$= 1.71$

Now, P(1.71) = 0 to be found with least degree and integer coefficient soon that gcd of all coefficient is 1.

Let x = 1.71

100x = 171

$\therefore$ P(x) = 100x – 171 = 0 is the polynomial which satisfied all the conditions.

$\therefore$ m = 100 + 171 = 271

$\therefore$ $\sqrt{m}$ = 16.46

$\therefore$ Nearest integer = 16

But this question can have multiple solutions as student can take tan 36 as 0.72, 0.726 or even 0.7. Every time we will get different answers. So this question should be Bonus.

24. For n ≥ 1, let a_n be the number beginning with n 9's followed by 744 ; e.g., a_4 = 9999744.
Define f(n) = max {m $\in$ N | 2^m divides a_n}, for n ≥ 1. Find f(1) + f(2) + f(3) +…. + f(10).

Sol. **(75)**

$\because$ $n \geq 1$

$a_1 = 9744$

$a_2 = 99744$

$a_3 = 999744$

$\vdots$

$a_{10} = \underbrace{999\ldots9}_{10\text{ times}}744$

$\because$ $f(n) = \max\left\{m \in \dfrac{N}{2^m} \text{ divides } a_n\right\}$

f (1) = 2^m divides 9744

$9744 = 2 \times 2 \times 2 \times 2 \times 609$

$= 2^4 \times 609$

9744 is divided by 2^4

$\therefore$ m = 4

f (2) = 2^m should divide 99744

$99744 = 2 \times 2 \times 2 \times 2 \times 2 \times 3117$

$99744 = 2^5 \times 3117$

a_2 = 99744 divided by 2^5

$\therefore$ m = 5

f (3) = 2^m should divide 999744

$999744 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 15621$

$999744 = 2^6 \times 15621$

a_3 = 999744 divided by 2^6

$\therefore$ m = 6

Similarly a_4 divided by 2^7

$\therefore$ m = 7

a_5 divides by 2^{13}

a_6 divides by 2^8

a_7 divides by 2^8

a_8 divides by 2^8

a_9 divides by 2^8

a_{10} divides by 2^8

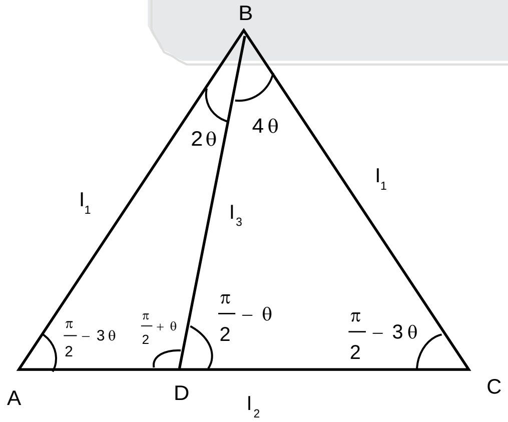
$\therefore$ f(1) + f(2) + f(3) ....+ f(10)

= 4 + 5 + 6 + 7 + 13 + 8 + 8 + 8 + 8 + 8

= 75

25. Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle$ B meets AC at D. If AB, AC and BD are integers and AB – BD = 3, find AC.

Sol. **(26)**



$$\frac{l_1}{\sin\left(\frac{\pi}{2} + \theta\right)} = \frac{l_3}{\sin\left(\frac{\pi}{2} - 3\theta\right)}$$

$\Rightarrow$ put $l_3 = l_1 - 3$ $\qquad \Rightarrow \quad l_1 = \frac{3}{4}\text{cosec}^2\theta$

If $\text{cosec}^2\theta = 36$

i.e. $\text{cosec}^2\theta = 6$

i.e. $\sin\theta = \frac{1}{6}$

$l_1 = \frac{3}{4} \times 36 = 27$

$l_3 = 24$

$\frac{l_1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{DC}{\sin 4\theta}$ & $\frac{l_1}{\sin\left(\frac{\pi}{2} + \theta\right)} = \frac{AD}{\sin 2\theta}$

$\Rightarrow \quad l_2 = \frac{l_1 \sin 4\theta}{\cos\theta} + \frac{l_1 \sin 2\theta}{\cos\theta} \qquad \Rightarrow \qquad l_2 = l_1 \times 2 \times \sin 3\theta$

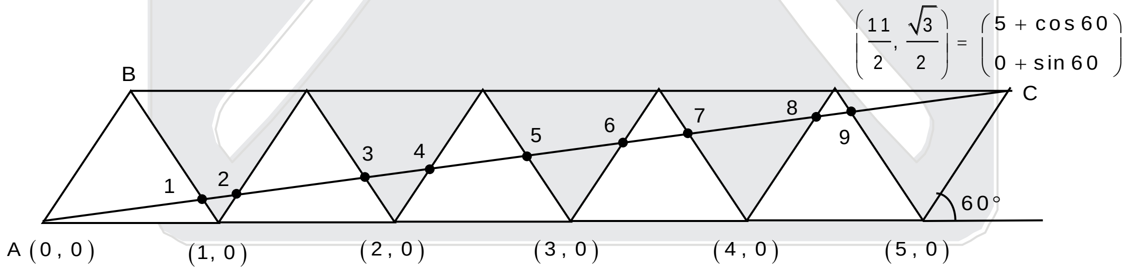
$\Rightarrow \quad \frac{3}{4}\text{cosec}^2\theta \times 2 \times \sin 3\theta \qquad \Rightarrow \qquad \frac{3}{2}\left(\frac{3}{\sin\theta} - 4\sin\theta\right)$

$l_2 = \frac{3}{2}\left(3 \times 6 - \frac{4}{6}\right) = \frac{3}{2}\left(18 - \frac{2}{3}\right) = \frac{3}{2}\left(\frac{54-2}{3}\right) = \frac{3}{2} \times \frac{52}{3}$

$l_2 = 26$

26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equal the angle of reflection. The distance traveled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N ?

Sol. **(31)**

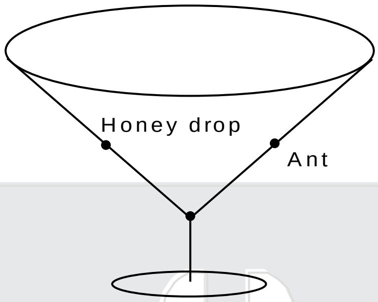


$AX = \sqrt{N}$

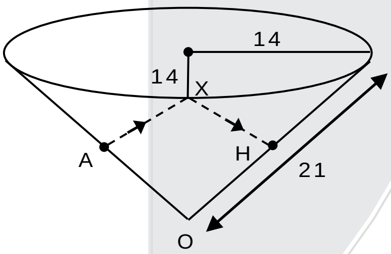
$AX^2 = \sqrt{\frac{121}{4} + \frac{3}{4}} = \sqrt{31}$

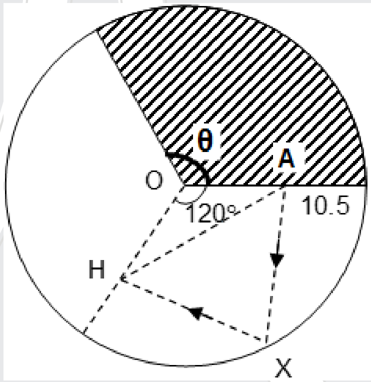
N = 31

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (see the figure). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d ? (Ignore the thickness of the glass)



Sol. (36)



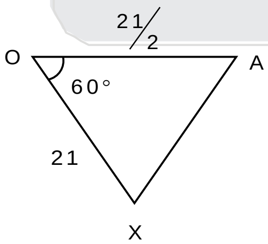


Assume we open the cone, it will become a section of a bigger circle with radius 21.

Section of the circle used to make cone = $\frac{2\pi \cdot 14}{2\pi \cdot 21} = \frac{2}{3}$ $(\because \theta = 120^\circ)$

$\therefore$ minimum distance 'd' = AX + XH = 2AX

In ΔDAX ,



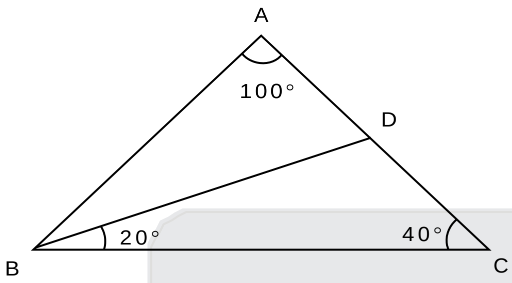
$$\text{Cos}60^\circ = \frac{(21)^2 + \left(\frac{21}{2}\right)^2 - (AX)^2}{2 \cdot 21 \cdot \frac{21}{2}} \Rightarrow AX = 21\left(\frac{\sqrt{3}}{2}\right)$$

$\therefore$ d = 36.37, [d] = 36

28. In a triangle ABC, it is known that $\angle$ A = 100° and AB = AC. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that sin 10° $\approx$ 0.174.

Sol. **(27)**

ΔBCD , by sine rule



$$\frac{BC}{120°} = \frac{BD}{\sin 40°} = \frac{20}{\sin 40°}$$

$\therefore$ BC = $\frac{10\sqrt{3}}{\sin 40°}$

Given, $\cos 80° (\sin 10°) \approx 0.174$

$$\sin 40° \approx \sqrt{\frac{1-\cos 80°}{2}} \approx 0.643$$

$\therefore$ Nearest integer to BC= 27

29. Let ABC be an acute angled triangle with AB =15 and BC = 8. Let D be a point on AB such that BD = BC. Consider points E on AC such that $\angle$ DEB = $\angle$ BEC. If α denotes the product of all possible value of AE, find [α] the integer part of α .

Sol. **(Bonus)**

The problem, as stated, has infinite solutions. If you take ANY triangle ABC with AB = 15 and BC= 8 construct point

D on AB such that BD = 8 and draw the bisector BE, then you have a perfectly valid triangle.

As the triangle is acute-angled, then AE can take any value between $\frac{15}{23}\sqrt{161} \approx \sqrt{8.275}$

(case when $\angle BCA$=90° and $\left(\frac{15}{23}\right)17$ ≈ 11.087

(case when $\angle ABC$=90°)

Question will have sense if only integer values for AE were allowed. If that is the case, then the possible integer values for AE are : 9, 10, 11

30. For any real number x, let [x] denotes the integer part of x; {x} be the fractional part of x ({x} = x − |x|). Let A denote the set of all real numbers x satisfying

$$\{x\} = \frac{x + [x] + \left[x + (1/2)\right]}{20}$$

If S is the sum of all numbers in A, find [S]

Sol. **(21)**

Let, x = I + f

∴ given equation reduces to,

$$f = \frac{I + f + I + I + \left[f + \frac{1}{2}\right]}{20}$$

(I) $f \in \left[0, \frac{1}{2}\right)$

$\left[f + \frac{1}{2}\right] = 0$

∴ $19f = 3I \Rightarrow f = \frac{3I}{19}$

$0, < \frac{3I}{19} < \frac{1}{2}$

⇒ $I \in \left[0, \frac{19}{6}\right) \Rightarrow$ I = 0, 1, 2, 3

∴ $f = 0, \frac{3}{19}, \frac{6}{19}, \frac{9}{19}$

∴ $x = 0, \frac{22}{19}, \frac{44}{19}, \frac{66}{19}$

(II) $f \in \left[\frac{1}{2}, 1\right)$

$\left[f + \frac{1}{2}\right] = 1$

$20f = 3I + 1 + f$

$f = \frac{3I + 1}{19}$

$\frac{1}{2} \le \frac{3I + 1}{19} < 1 \Rightarrow \frac{17}{6} \le I < 6 \Rightarrow$ I = 3, 4, 5

∴ $f = \frac{10}{19}, \frac{13}{19}, \frac{16}{19}$

∴ $x = \frac{67}{19}, \frac{89}{19}, \frac{111}{19}$

∴ $S = \frac{399}{19}$

[S] = 21