ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA (AMTI)

JUNIOR LEVEL-2020-21

Classes IX & X

Saturday, 24th April 2021

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. The first five terms of a sequence are 1,2,3,4 and 5, From the sixth term on, each term is 1 less than the product of all the preceding ones, Prove that the product of the first 70 terms is equal to the sum of their squares.

Sol. $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, $T_4 = 4$, $T_5 = 5$

$$\left.\begin{array}{l} T_6 = 1\times2\times3\times4\times5-1=119 \\ \&1^2+2^2+3^2+4^2+5^2=55 \end{array}\right] - \text{diff} = 64$$

$$\left.\begin{array}{l} T_7 = 119\times5\times4\times3\times2\times1-1=14279 \\ \&\ \ 119^2+1^2+2^2+3^2+4^2+5^2=14216 \end{array}\right] - \text{diff} = 63$$

$$\left.\begin{array}{l} T_8 = 14279\times119........\times2\times1=203904119 \\ \&\ \ 14279^2+119^2+5^2+.......2^2+1^2=203904057 \end{array}\right] - \text{diff} = 62$$

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So,

$T_{71} = [T_{70}T_{69} \times .... \times 3 \times 2 \times 1 - 1] - \left[T_{70}^2+T_{69}^2+.......+2^2+1^2\right] = -1$

So,

$T_{70} \times T_{69} \times .... \times 3 \times 2 \times 1 = T_{70}^2+T_{69}^2+.......+2^2+1^2$ Hence proved

2. The number N equals the product of 100 different positive integers, Show that N has at least 4951 different divisors (including 1 and the number itself). What kind of numbers N will have exactly 4951 divisors, if such numberi exist ?

Sol. N is product of 100 positive integers.

$N = a_1, a_2, a_3, a_4, a_5 .... a_{100}$

For least number of factors of N all positive integers should be same prime number with different powers.

$N = 1 \times P^1 \times P^2 \times P^3 ..... P^{99}$

$N = 1 \times P^{1+2+3 ....99} = P^{\frac{99\times100}{2}}$

$= P^{4950}$

So least number of factors (f) = (4950 + 1) =4951

Hence f ≥ 4951

For number of factors f = 4951
N should be of form :
$N = P \times P^2 \times P^3 \times P^4 \ldots\ldots P^{99}$
here N is a prime number

3. Find all positive integers k such that the product of the decimal digits of k equals $\frac{25}{8}k - 211$.

Sol. K is a positive integer and product of digits is given by –

$$P = \frac{25K}{8} - 211$$

Product of digits of number will be positive integer.
for that k must be multiple of 8.

at	k = 5, 16, 24 …….. 64	P < 0	
at	k = 72	P = 14 = 7 × 2	✓
at	k = 80	P = 39	×
at	k = 88	P = 64 = 8 × 8	✓
at	k = 96	P = 89	×
at	k = 104	P = 114	×
at	k = 112	P = 129	×
at	k = 120	P = 164	×

Now if we increase value of k product increase more faster. So for larger values of k it is not possible that

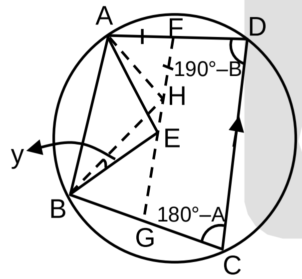
$$P = \frac{25K}{8} - 211$$

Hence only two values of k = 72 and 88 are possible.

4. Given three non-collinear points A, B, C, construct a circle with centre C such that the tangents from A and B to the circle are parallel.

5. Let ABCD be a cyclic quadrilateral where E is the point of intersection of the angle bisectors of ∠A and ∠B. Let a line through CD meet AD, BC at F and G respectively. Prove that FG = AF + BG.

Sol.



Take a point H on FG such that
AF = FH ...(1)
∠HAE = ∠EAF – ∠FAH
∠AFH = ∠FDC = 180 – B
In ΔAFH –

$$\angle FAH = \angle FHA = \frac{B}{2}$$

So, $\angle HAE = \frac{A}{2} - \frac{B}{2} = \frac{A-B}{2}$

Let $\angle HBE = y$

$$\angle ABH = \frac{B}{2} - y$$

Now, In ΔAHB :

$$\left(\frac{A}{2} + \frac{A-B}{2}\right) + (\angle AHB) + \left(\frac{B}{2} - y\right) = 180$$

$A - \frac{B}{2} + \angle AHB + \frac{B}{2} - y = 180$

$\angle AHB = 180 - A + y$

Now $\angle BHG = 180 - \left\{\frac{B}{2} + 180 - A + y\right\}$

$= 180 - \frac{B}{2} - 180 + A - y$

$= A - \frac{B}{2} - y$

In Quadrilateral ABEH :

$\angle ABE + \angle AHE =$

$= \frac{B}{2} + 180 - \frac{B}{2} = 180$

So ABEH is Cyclic

So $\angle HBE = \angle EAH$

$y = \frac{A - B}{2}$

Now in ΔBHG :

$\angle HBG = y + \frac{B}{2} = \frac{A-B}{2} + \frac{B}{2} = \frac{A}{2}$

$\angle HGB = 180 - A$

$\angle BHG = 180 - \left[180 - A + \frac{A}{2}\right] = \frac{A}{2}$

$\therefore \angle HBG = \angle BHG$

So BG = HG ... (2)

From equation (1) and (2)

AF + BG = FH + HG

= FG

6. Find all positive integers n<2021 whose square ends with 444.

Sol. Let $(abcd)^2 = ...\ 444$

$(1000a + 100b + 10c + d)^2$

$= (10000a)^2 + (100b)^2 + (10c)^2 + (d)^2$

$+ 200000ab + 20000bc + 2000ad$

$+ 2000bc + 200bd + 20cd = ...\ 444$

By taking mod 1000

$100(c^2 + 2bd) + 20cd + d^2 = 444$

d = 2 or 8

Case 1 : when d = 2

$100(c^2 + 4b) + 40c = 440$

$10(c^2 + 4b) + 4c = 44$

C = 1 or 6

When C = 1

$10[1 + 4b] + 4 = 44$

4b = 3 (Not possible)

When C = 6

$10[36 + 4b] + 24 = 44$

Possible value of b = 4, 9

Case 2: When d = 8

$100(c^2 + 16b) + 160c + 64 = 444$

$10[10(c^2 + 6b) + 16c)] = 380$

C = 3 or 8

When c =3

$10(9 + 16b) + 48 = 38$

$9 + 16b + 1 = 0$

$b = 0$ or 5

When c = 8

$10[64 + 16b] + 16 \times 8 = 38$

$10(64 \_ 16b) + 128 = 38$

$16b + 73 = 0$ Not possible

So, Possible values of (b,cd)

= (4, 6, 2), (9, 6, 2), (0, 3, 8), (5, 3, 8)

$\because n < 2021$

When $a = 0 \Rightarrow$ 4 Numbers

(462, 962, 038, 538)

When $a = 1 \Rightarrow$ 4 Numbers

(1462, 1962, 1038, 1538)

Total 8 Numbers are possible.

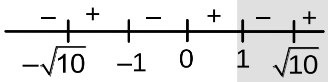
7. Prove, for any real number c, the equation $x(x^2-1)(x^2-10) = c$ cannot have five integer solutions.

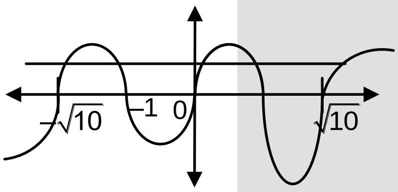
Sol. Given that

$x(x^2-1)(x^2-10) = C$

Let $f(x) = x(x^2-1)(x^2-10)$

$= x(x-1)(x+1)(x-\sqrt{10})(x-\sqrt{10})$





From graph we can see that when any line y = c cuts graph at five points then values of x lies between $-\sqrt{10}$ to $\sqrt{10}$.

x can take integer values –3, –2, –1, 0, 1, 2, 3

at x = 3 f(x) = 24

x = –3 f(x) = –24

x = 2 f(x) = 36

x = –2 f(x) = –36

x = 1 f(x) = 0

x = –1 f(x) = 0

x = 0 f(x) = 0

There is not any value of f(x) which is same for five different integer values of x.

So we can say that

$x(x^2 - 1)(x^2 - 10) = C$ have not five integer solution.

8. The real numbers α and β are such that $\alpha^3 - 3\alpha^2 + 5\alpha = 1$ and $\beta^3 - 3\beta^2 + 5\beta = 5$ are satisfied. Find $\alpha + \beta$.

Sol. Given that :

$\alpha^3 - 3\alpha^2 + 5\alpha = 1$ …(1)

$\beta^3 - 3\beta^2 + 5\beta = 5$ …(2)

Add eq. (1) and (2) :

$\alpha^2+\beta^3 - 3(\alpha^2+\beta^2) + 5(\alpha+\beta) = 6$

$[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)] - 3[(\alpha+\beta)^2 - 2\alpha\beta] + 5(\alpha+\beta) = 6$

$(\alpha+\beta)^3 + (\alpha+\beta)(5-3\alpha\beta) - 3(\alpha+\beta)^2 + 6\alpha\beta - 6 = 0$

= Let $\alpha+\beta = a$ and $3-3\alpha\beta = b$

$a^3 + a(2+b) - 3a^2 - 2b = 0$

$a^3 - 3a^2 + a(2+b) - 2b = 0$

a=2 Satisfy this equation so (a–2) is a factor of this equation.

$(a-2)(a^2-a+b) = 0$

$a = 2$

$\alpha+\beta = 2$

Hence $\alpha+\beta = 2$