GAUSS CONTEST - FINAL - PRIMARY
Classes V & VI
AMTI - Saturday, 2nd November_2019.

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. **(a)** Find the positive integers m, n such that $\frac{1}{m}+\frac{1}{n}=\frac{3}{17}$.

(b) Find the positive integers m, n, p such that $\frac{1}{m}+\frac{1}{n}+\frac{1}{p}=\frac{3}{17}$.

Sol.

(a) We know

If $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$

then, $(x – a)(y – a) = a^2$

so, $\frac{1}{m}+\frac{1}{n}=\frac{3}{17}$

$\frac{1}{3m}+\frac{1}{3n}=\frac{1}{17}$

$(3m – 17)(3n – 17) = 289$
$= 289 \times 1$
$= 17 \times 17$
$= 1 \times 289$

If $(3m – 17) = 289$ and $3n – 17 = 1$

$m = 102$ $\quad n = 6$

so, $(m, n) = (102, 6)$

If $3m – 17 = 17$ and $3n – 17 = 17$

$m = \frac{34}{3}$ $\quad n = \frac{34}{3}$

not integer so reject

If $(3m – 17) = 1$ and $3n – 17 = 289$

$m = 6$ $\quad n = 102$

so, $(m, n) = (6, 102)$

so, $\frac{1}{6}+\frac{1}{102}=\frac{3}{17}$ ....(i)

(b) Now,

If $\frac{1}{6}=\frac{1}{x}+\frac{1}{y}$

$(x – 6)(y – 6) = 36$
$= 1 \times 36$ or 36×1
$= 2 \times 18$ or 18×2
$= 3 \times 12$ or 12×3
$= 4 \times 9$ or 9×4
$= 6 \times 6$

so, (x, y) = (7,42) (8, 24), (9, 18), (10, 15) (12, 12)

so from equation (i)

$$\frac{1}{7}+\frac{1}{42}+\frac{1}{102}=\frac{3}{17}$$

$$\frac{1}{8}+\frac{1}{24}+\frac{1}{102}=\frac{3}{17}$$

: :

: :

and $\frac{1}{102}=\frac{1}{w}+\frac{1}{z}$

$(w – 102) (z – 102) = (102)^2$

= 1 × 10404

= 2 × 5202

= : :

= : :

= 102 × 102

Total 27 in which 13 are repeated so total 14 different pais.

so pairs of (w, z) = (103, 10506), (104, 5304).......... (204, 204)

so total 14 pairs

from equation (i)

$$\frac{1}{6}+\frac{1}{103}+\frac{1}{10506}=\frac{3}{17}$$

$$\frac{1}{6}+\frac{1}{104}+\frac{1}{5304}=\frac{3}{17}$$

: :

: :

Total 5 + 14 = 19 pairs.

2. Find the largest positive integer n such that 3^n divides the 999 digit number 9999....99.

Sol. For number $\underbrace{999.......9}_{999\text{ times}}$

sum of digits = 999 × 9

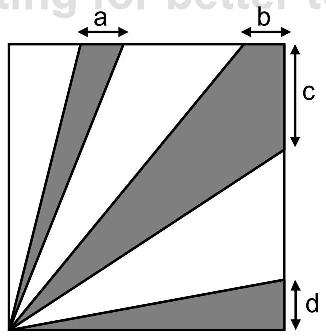
= 9 × 9 × 111

= 3^5 × 37

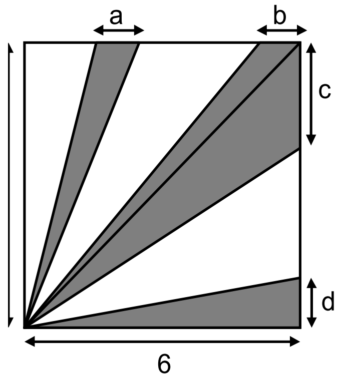
so it is divisible by 3^5

largest value of n = 5.

3. Inside a square of area 36 cm^2, there are shaded regions as shown. The ratio of the shaded area to the unshaded area is 3 : 1. What is the value of a + b + c + d where a, b, c, d are the lengths of the bases of the shaded regions ? Further, if three of a, b, c, d are equal integers and one different, then find them.



Sol.



Shaded area = $\frac{1}{2} \times 6 \times a + \frac{1}{2} \times 6 \times b + \frac{1}{2} \times 6 \times c + \frac{1}{2} \times 6 \times d$

3(a + b + c + d)

So unshaded area = 36 – 3 (a + b + c + d)

A.T.Q.

$$\frac{3(a+b+c+d)}{36-3(a+b+c+d)} = \frac{3}{1}$$

(a + b + c + d) = 36 – 3(a + b + c + d)

4(a + b + c + d) = 36.

$\Rightarrow$ a + b + c + d = 9

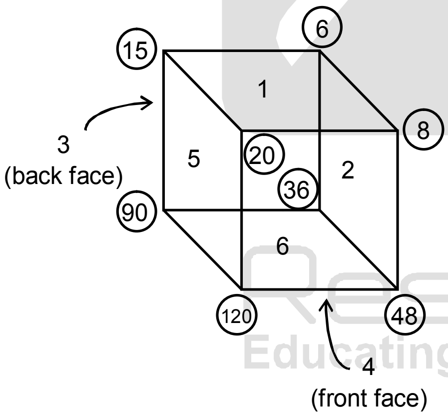
If a = b = c = 1 then d = 6 (not possible)

If a = b = c = 2 then d = 3 (possible)

so equal three integers are 2 and different is 3.

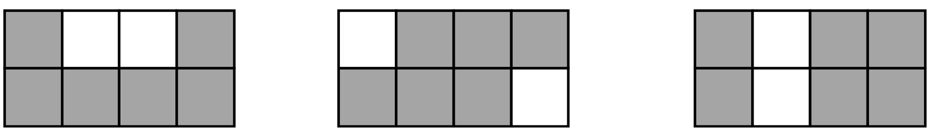
4. Let the six faces of a cube be numbered 1, 2, 3, 4, 5, 6 in such a way that the 3 pairs (1, 6), (2, 5), (3, 4) lie on opposite faces of the cube. At each vertex of the cube, the product of the three numbers on the three faces containing the vertex is written. What is the sum of all the eight numbers written at the eight vertices of the cube ?

Sol.



Sum of numbers = 15 + 6 + 20 + 8 + 90 + 36 + 120 + 48 = 343.

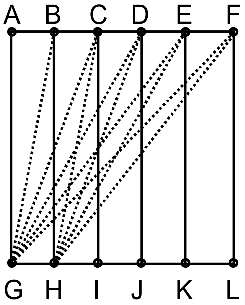
5. Given a 2 × 4 rectangle with eight cells, find the total number of ways (frames) in which you can shade 75% of the cells. Few such frames are given below.



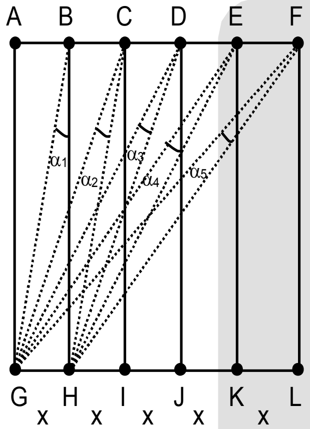
Sol. 75% shaded means = 6 cells are shaded.

so selecting 6 cells from 8 cells is = ${}^{8}C_6 = \frac{8 \times 7}{2 \times 1} = 28$.

6. A square is divided into 5 identical rectangles as in the figure. Find the sum of the angles ∠GBH, ∠GCH, ∠GDH, ∠GEH, ∠GFH. Given a valid proof for your answer.



Sol.



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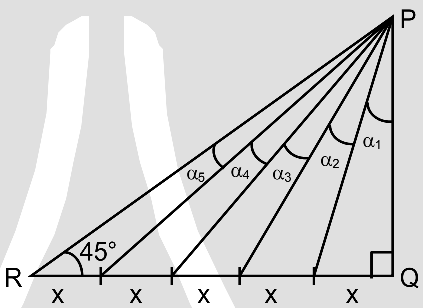


PQ = BH

PR = FG

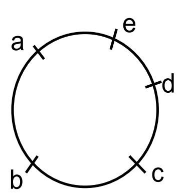
= ∠PRQ = ∠FGL

∠G = 45°

∠RPQ = $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 45°$.

7. Around a circle five positive integers a, b, c, d, e are written in such a way that the sum of no three or no two adjacent integers is divisible by three. How many of these a, b, c, d, e are divisible by three ? Please given proper proof for your answer.

Sol.

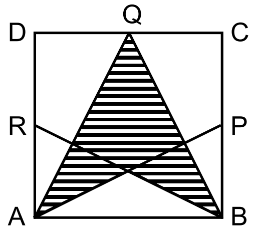


Possibilies of a, b, c, d, e

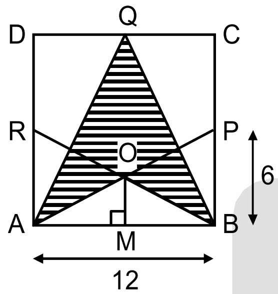
a	b	c	d	e	
3k	3k + 1	3k + 1	3k	3k + 1	(possible)
3k	3k + 1	3k	3k + 1	3k	(a + e = 3k not possible)
3k	3k + 2	3k	3k + 2	3k	(a + e = 3k not possible)
3k	3k + 2	3k + 2	3k	3k + 2	(possible)

So two numbers among a, b, c, d, and e is divisible by three.

8. Let ABCD be a square with the length of side equal to 12 cm. Points P, Q, R are respectively the midpoints of side BC, CD and DA respectively (see figure). Find the area of the shaded region in square cm. Given valid explanation for your steps.



Sol.



ar(ABCD) = 144 cm^2

ABPR is a rectangle

$\therefore$ AO = OP = BO = OR

O is the mid point of AP

OM $\perp$ AB

Hence OM || PB

M is also mid point of AB

OM = $\frac{1}{2}$PB (mid point theorem) = $\frac{1}{2}$ × 6 = 3cm

ar(ΔAOB) = $\frac{1}{2}$ × 12 × 3 = 18 cm^2

ar(ΔABQ) = $\frac{1}{2}$ ar (ABCD) = $\frac{1}{2}$ × 144 = 72

So shaded area = ar(ΔABQ) – ar(ΔAOB).

= 72 – 18 = 54 cm^2.