THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Screening Test - Bhaskara Contest

NMTC at JUNIOR LEVEL - IX & X Standards

Saturday, 31 August, 2019

Note:

1. Fill in the response sheet with your Name, Class and the institution through which you appear in the specified places.

2. Diagrams are only visual aids; they are NOT drawn to scale.

3. You are free to do rough work on separate sheets.

4. Duration of the test: **2 hours.**

PART—A

Note

- Only one of the choices A. B, C, D is correct for each question. Shade the alphabet of your choice in the response sheet. If you have any doubt in the method of answering, seek the guidance of the supervisor.
- For each correct response you get 1 mark. **For each incorrect response you lose $\frac{1}{2}$ mark.**

1. The number of 6 digit numbers of the form "ABCABC", which are divisible by 13, where A, B and C are distinct digits, A and C being even digits is

(A) 200 (B) 250 (C) 160 (D) 128

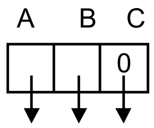
Sol. **(D)**

1001 × ABC = ABCABC

where 1001 = 13 × 7 × 11

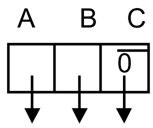
Now A and C are even digits and A, B, C are different digits

Case-I : When C is zero



4 × 8 × 1 = 32

Case-II : When C is not zero



4 × 8 × 3 = 96

Total number of 6 digits

Number possible = 32 + 96 = 128

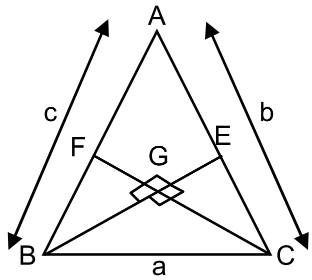
Option (D).

2. In ΔABC, the medians through B and C are perpendicular. Then $b^2 + c^2$ is equal to

(A) $2a^2$ (B) $3a^2$ (C) $4a^2$ (D) $5a^2$

Sol. **(D)**

Let BG = 2x, GE = x



CG = 2y, GF = y

In ΔGCE

$$(2y)^2 + x^2 = \left(\frac{b}{2}\right)^2$$

$$4y^2 + x^2 = \frac{b^2}{4} \qquad \text{...........(i)}$$

In ΔBCG

$$(2x)^2 + y^2 = \left(\frac{c}{2}\right)^2$$

$$4x^2 + y^2 = \frac{c^2}{4} \qquad \text{............(ii)}$$

In ΔBGE

$$(2x)^2 + (2y)^2 = a^2$$

$$4(x^2 + y^2) = a^2$$

$$x^2 + y^2 = \frac{a^2}{4} \qquad \text{.......(iii)}$$

Equation (i) + (ii)

$$5x^2 + 5y^2 = \frac{b^2 + c^2}{4}$$

$$5(x^2 = y^2) = \frac{b^2 + c^2}{4}$$

from equation (iii)

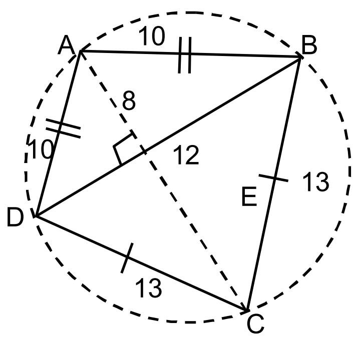
$$5\left(\frac{a^2}{4}\right) = \frac{b^2 + c^2}{4} \quad \Rightarrow \quad b^2 + c^2 = 5a^2$$

Option (D).

3. In a quadrilateral ABCD, AB = AD = 10, BD = 12, CB = CD = 13. Then

(A) ABCD is a cyclic quadrilateral
(B) ABCD has an in-circle
(C) ABCD has both circum-circle and in-circle
(D) It has neither a circum-circle nor an in-circle

Sol. **(B)**



$AM = \sqrt{10^2 - 6^2} = 8$

$CM = \sqrt{13^2 - 6^2} = \sqrt{133}$

For incircle

AB + DC = AD + BC

23 = 23

In circle is possible

for cyclic quadrilateral (circumcircle) theorem should be followed.

$AC \times BD = AB \cdot CD + BC \cdot AD$

$\left(8+\sqrt{133}\right) \times 12 \neq 10 \times 13 + 10 \times 13$

It is not a cyclic quadrilateral

Option (B).

4. Given three cubes with integer side lengths, if the sum of the surface areas of the three cubes is 498 sq. cm, then the sum of the volumes of the cubes in all possible solutions is

(A) 731 (B) 495 (C) 1226 (D) None of these

Sol. **(C)**

$6(x^2 + y^2 + z^2) = 498$

$x^2 + y^2 + z^2 = 83$

for x, y, z to be integer

$x = \sqrt{49}$, $y = \sqrt{25}$, $z = \sqrt{9}$

x = 7, y = 5, z = 3

Sum of volumes = $7^3 + 5^3 + 3^3$

$\Rightarrow$ 343 + 125 + 27 = 495

for x, y, z to be integer

$x = \sqrt{81}$, $y = \sqrt{11}$, $z = \sqrt{1}$

x = 9, y = 1, z = 1

Sum of volumes = $9^3 + 1^3 + 1^3$

= 729 + 1 + 1 = 731.

So, total sum = 495 + 731 = 1226

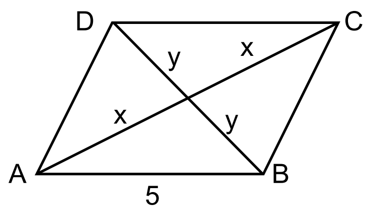
Option (C).

5. In a rhombus of side length 5, the length of one of the diagonals is at least 6, and the length of the other diagonal is at most 6. What is the maximum value of the sum of the diagonals ?

(A) $10\sqrt{2}$ (B) 14 (C) $5\sqrt{6}$ (D) 12

Sol. **(B)**

Let diagonal are 2x and 2y



$x^2 + y^2 = 25$

We have to find $2(x + y)_{max}$ = ?

$2x \geq 6$ $2y \leq 6$

$x \geq 3$ $y \leq 3$

By option (A) $2(x + y) = 10\sqrt{2}$

$x^2 + y^2 = 25$

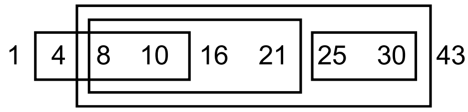
from here we get $x = y = \frac{5}{\sqrt{2}}$ it is not possible.

$2y = 7.070 \geq 6$.

By option (B)
$2(x + y) = 14$
$x^2 + y^2 = 25$
$2y = 6$ and $2x = 8$
it is possible maximum value which is greater by other two options.

6. In the sequence 1, 4, 8, 10, 16, 21, 25, 30 and 43, the number of blocks of consecutive terms whose sums are divisible by 11 is
(A) only one (B) exactly two (C) exactly three (D) exactly four

Sol. **(D)**



$4 + 8 + 10 = 22$
$8 + 10 + 16 + 21 = 55$
$8 + 10 + 16 + 21 + 25 + 30 = 110$
$25 + 30 = 55$
Option (D).

7. Let A = {1, 2, 3,............, 17}. For every nonempty subset B of A find the product of the reciprocals of the members of B. The sum of all such product is
(A) $\frac{153}{17!}$ (B) $\frac{153}{\text{lcm}(1, 2, ...,17)}$ (C) 18 (D) 17

Sol. **(D)**

$$\left(\frac{1}{1}+\frac{1}{2}........\frac{1}{17}\right)+\left(\frac{1}{1\times 2}+\frac{1}{1\times 3}\right)+......\left(\frac{1}{1\times 2\times 3.....17}\right)$$

$$= \frac{(1+2+3........17)+(1\times 2+1\times 3+...)......+(1\times 2......16)+1}{1\times 2\times 3.....17}$$

$$= \frac{\Sigma\, 1+\Sigma\, 1.2+\Sigma\, 1.2.3+.....+\Sigma 1.2......16+1}{1\times 2\times 3....17}$$

$$= \frac{(1+1)\,(1+2)\,(1+3)........(1+17)-(1\times 2....17)}{1\times 2\times 3....17}$$

$$= \frac{1.\,2\,.\,3..........18-1.2.3.......17}{1\times 2\times 3....17} = \frac{17!\,(18-1)}{17!} = 17.$$

8. The remainder of $f(x) = x^{100} + x^{50} + x^{10} + x^2 - 6$ when divided by $x^2 - 1$ is
(A) x + 1 (B) – 2 (C) 0 (D) 2

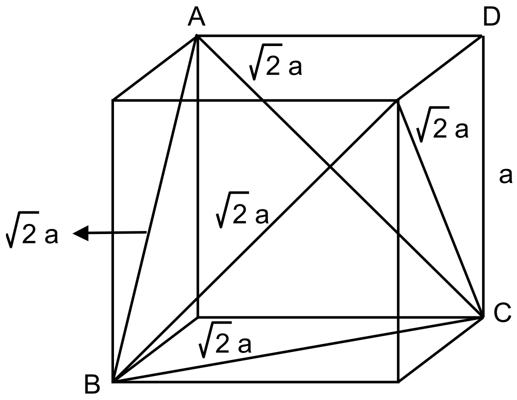
Sol. **(B)**
Let $R(x) = Ax + B$
$x^{100} + x^{50} + x^{10} + x^2 - 6 = q(x)\,(x^2 - 1) + Ax + B$
$x = 1$
$1 + 1 + 1 + 1 - 6 = A + B$
$-2 = A + B$ ....(i)
$x = -1$
$1 + 1 + 1 + 1 - 6 = -A + B$
$-2 = -A + B$ ....(ii)
from equation (i) and (ii)
$-4 = 2B$
$B = -2$
$A = 0$
$R(x) = -2$
Option (B).

9. The number of acute angled triangles whose vertices are chosen from the vertices of a rectangular box is

(A) 6 (B) 8 (C) 12 (D) 24

Sol. **(B)**

From each surface diagonals there are 4 triangles are possible which are equilateral of side $\sqrt{2}a$ each.



But in this manner each triangle is counted thrice therefore $\frac{24}{3} \Rightarrow 8$.

Option (B).

10. In the subtraction below, what is the sum of the digits in the result ?

111...........111 (100 digits) – 222.......222 (50 digits)

(A) 375 (B) 420 (C) 429 (D) 450

Sol. **(D)**

1111..................111...........111

222...........222

1111..................108............889

49 times 1, 49 times 8 and 1 times 0 and 9

Sum = 49 × 1 + 8 × 49 + 9

$\Rightarrow$ 49 + 392 + 9 = 450.

Option (D).

11. If m and n are positive integers such that $\frac{m+n}{m^2+mn+n^2} = \frac{4}{49}$, then m + n is equal to

(A) 4 (B) 8 (C) 12 (D) 16

Sol. **(D)**

$\frac{m+n}{m^2+mn+n^2} = \frac{4}{49}$

Let m + n = 4k and $m^2 + mn + n^2 = 49k$

$m^2 + mn + n^2 = 49k$

$(m + n)^2 – mn = 49k$

$(4k)^2 – mn = 49k$

$mn = 16k^2 – 49k$

$= k(16k – 49)$

m and n are positive integer so k and 16k – 49 is also positive.

16k – 49 > 0

$k > \frac{49}{16}$

k > 3

So, $\frac{m+n}{4} > 3$

m + n > 12

from options m + n = 16

Alternate :

$\frac{m+n}{m^2+mn+n^2} = \frac{4}{49}$

$\frac{m+n}{(m+n)^2-mn} = \frac{4}{49}$

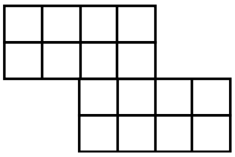
m and n are positive integer mn > 0

from option a, b, c

mn < 0

therefore m + n = 16.

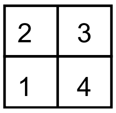
12. Given a sheet of 16 stamps as shown, the number of ways of choosing three connected stamps (two adjacent stamps must have an edge in common) is



(A) 40 (B) 41 (C) 42 (D) 44

Sol. **(C)**

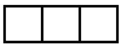
In such block there are 4 such combination (1, 2, 3) (2, 3, 4) (3, 4, 1), (4, 1, 2)



So there 7 such blocks

So total combination = 7 × 4 = 28.

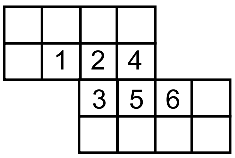
In such block there in 1 such combination.



There are 12 such block.

so total combination = 12.

(1, 2, 3), (4, 5, 6) are 2 more combination available.



So finally total combination = 28 + 12 + 2 = 42.

13. In an election 320 votes were cast for five candidates. The winner's margins over the other four candidates were 9, 13, 18 and 25. The lowest number of votes received by a candidate was

(A) 49 (B) 50 (C) 51 (D) 52

Sol. **(D)**

Let winner get x votes other will get x – 9, x – 13, x – 18, x – 25

5x – 65 = 320

$5x \Rightarrow 385$

$x \Rightarrow 77$

Lowest number of votes = x – 25 = 77 – 25 = 52.

Option (D).

14. A competition has 25 questions and is marked as follows

(a) Five marks are awarded for each correct answer to questions 1 to 15

(b) Six marks are awarded for each correct answer to questions 16 to 25

(c) Each incorrect answer to questions 16 to 20 loses 1 mark

(s) Each incorrect answer to questions 21 to 25 loses 2 marks

(A) 126 (B) 127 (C) 128 (D) 129

Sol. **(A)**

Total marks = 135

129 is possible when he does not attempt one question of 6 marks 128 is possible when he attempt wrongly one question of 6 of 6 mark with one negative mark.

127 is possible when he attempt wrongly one question of 6 mark with 2 negative mark.

Option (A)

15. A, M, T, I are positive integers such that A + M + T + I = 10. The maximum possible value of A × M × T × I + A × M × T + A × M × I + A × T × I + M × T × I + A × M + A × T + A × I + M × T + M × I + T × I is

(A) 109 (B) 121 (C) 133 (D) 144

Sol. **(C)**

A × M × T × I + A × M × T + A × M × I + A × T × I + M × T × I + A × M + A × T + A × I + M × T + M × I + T × I this expression is maximum if we take A = M = 3, T = I = 2.

A × M × T × I + A × M × T + A × M × I + A × T × I + M × T × I + A × M + A × T + A × I + M × T + M × I + T × I = (1 + A) (1 + M) (1 + T) (1 + I) – 1 – (A + M + T + I)

= (1 + 3) (1 + 3) (1 + 2) (1 + 2) – 1 – 10

= 144 – 11 = 133.

Option (C).

PART - B

Note :

- Write the correct answer in the space provided in the response sheet
- For each correct response you get 1 mark. For each incorrect response you lose $\frac{1}{4}$ marks.

16. The three digit number XYZ when divided by 8, gives as quotient the two digit number ZX and remainder Y. The number XYZ is __________.

Sol. **(435)**

xyz = 8(10z + x) + y

100x + 10y + z = 80z + 8x + y

92x + 9y = 79z

9y = 79z – 92x

9y = 72z + 7z – 90x – 2x

$y = \frac{9(8z-10x)}{9} + \frac{7z-2x}{9}$

7z – 2x should be multiple of 9

z = 5, x = 4, y = 3.

xyz = 435.

17. The digit sum of any number is the sum of its digits. N is a 3 digit number. When the digit sum of N is subtracted from N, we obtain the square of the digit sum of N. The number N is _________.

Sol. **(156)**

Let s be the sum of digit of N (3 digit number)

$N - s = s^2$

$N = s^2 + s$

As maximum sum of 3 digit number is 27.

so we put the value of s upto 27 and check.

we observe if we put s = 12

$N = 12^2 + 12 = 156$ is the required number.

18. A 4 × 4 anti-magic square is an arrangement of the numbers 1 to 16 in a square so that the totals of each of the four rows, four columns and the two diagonals are ten consecutive numbers in some order. The diagram shows an incomplete anti magic square. When it is completed, the number in the position of * is ______________.

			14
*	9	3	7
	12	13	5
10	11	6	4

Sol. **(16)**

	C_1	C_2	C_3	C_4
R_1				14
R_2	*	9	3	7
R_3		12	13	5
R_4	10	11	6	4

D_1 D_2

C_4 sum = 30

R_4 sum = 31

D_1 sum = 39

as the sum of 4 column, 4 row and 2 diagonal is 10 consecutive integer as C_4 sum is 30 and D_1 sum is 39.

$\therefore$ the ten sum from 30 to 39.

So the remaining sum left is 32 to 38 and number which is to filled 1, 2, 8, 15, 16 only.

Sum of D_2 element $9 + 13 + 4 = 26 + \left\{\begin{matrix}1\\2\\8\\15\\16\end{matrix}\right\} = \left\{\begin{matrix}27\\28\\34\\41\\42\end{matrix}\right\}$

$\therefore$ In D_2 we filled 8.

Sum of R_3 element 12 + 13 + 5 = 30

$$30 + \left\{\begin{matrix}1\\2\\15\\16\end{matrix}\right\} = \left\{\begin{matrix}31\\32\\45\\46\end{matrix}\right\}$$

$\therefore$ In R_3 we fill 2.

Sum of C_2 element 9 + 12 + 11 = 32

$$32 + \left\{\begin{matrix}1\\15\\16\end{matrix}\right\} = \left\{\begin{matrix}33\\47\\48\end{matrix}\right\}$$

$\therefore$ In C_2 we filled 1.

Sum of element R_2

9 + 3 + 7 = 19

$$19 + \left\{\begin{matrix}15\\16\end{matrix}\right\} = \left\{\begin{matrix}34\\35\end{matrix}\right\}$$

$\therefore$ In R_2 we fill = 16

$\therefore$ In C_3 we fill = 15.

19. An escalator moves up at a constant rate. John walks up the escalator at the rate of one step per second and reaches the top in twenty seconds. The next day John's rate was two steps per second, and he reached the top in sixteen seconds. The number of steps in the escalator is ___________.

Sol. **(80)**

Let the speed of escalator = x steps/seconds.

Number of steps = $(x + 1) \times 20 = (x + 2) \times 16$

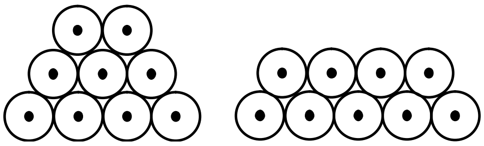
$20x + 20 = 16x + 32$

$4x = 12$

$x = 3.$

Number of steps = $(3 + 1) \times 20 = 80.$

20. In a stack of coins, each row has exactly one coin less than the row below. If we have nine coins, two such towers are possible. Of these, the tower on the left is the tallest. If you have 2015 coins, the height of the tallest towers is ____________.

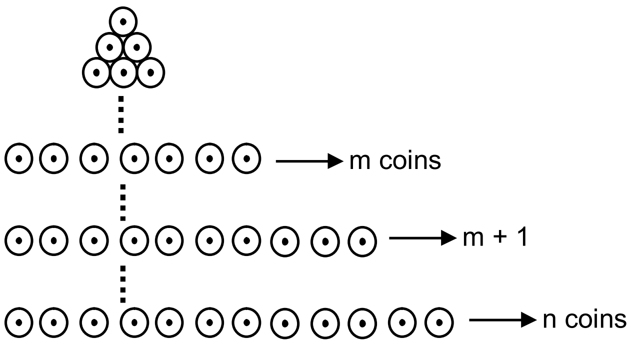


Sol. **(BONUS)**

As the radius of the coin is not given.

Let number of coins in last row of tower is n and (m + 1) coins in top row,

then we have to find $(n - m)_{max}$



$(1 + 2 + 3.....n) - (1 + 2 + 3 ....m) = 2015$

$$\frac{n(n+1)}{2} - \frac{m(m+1)}{2} = 2015$$

$n^2 + n - m^2 - m = 4030$

$(n - m)(n + m) + n - m = 4030$

$(n - m)(m + n + 1) = 4030$

1	4030
2	2015
5	806
13	310
31	130
10	403
26	155
62	65

$(n + m + 1) > n - m$

If we take $n - m = 62$ and $m - n + 1 = 65$

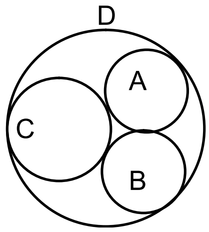
We get n = 63 and m = 1

$n - m = 62$

So $(n - m)_{max.} = 62$

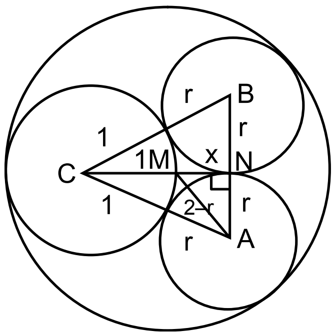
Height = Number of rows = 62.

21. Circles A, B and C are externally tangent to each other and internally tangent to circle D. Circles A and B are congruent. Circle C has radius 1 unit and passes through the centre of circle D. Then the radius of circle B is _________ units.



Sol. $\left(\frac{8}{9}\right)$

In ΔMAN



$(2-r)^2 = x^2 + r^2$

$4 + r^2 - 4r = x^2 + r^2$

$4(1-r) = x^2 \quad \Rightarrow \quad 4 - 4r = x^2 \quad \Rightarrow \quad r = \frac{4-x^2}{4}$

In ΔCAN

$(1 + x)^2 + r^2 = (1 + r)^2$

$1 + x^2 + 2x + r^2 = 1 + r^2 = 2r$

$x^2 + 2x = 2r$

$x^2 = 2r - 2x$

$\Rightarrow \quad x^2 = 2\left(\frac{4-x^2}{4}\right) - 2x$

$\Rightarrow \quad x^2 = \frac{4-x^2}{2} - 2x$

$\Rightarrow \quad 2x^2 = 4 - x^2 - 4x$

$3x^2 + 4x - 4 = 0$

$3x^2 + 6x - 2x - 4 = 0$

$3x(x + 2) - 2(x + 2) = 0$

$\Rightarrow \quad x = \frac{2}{3}, x = -2.$

$$r = 4 - \frac{\left(\frac{2}{3}\right)^2}{4} = \frac{4-\frac{4}{9}}{4} \Rightarrow \frac{36-4}{36} = \frac{32}{36} \Rightarrow \frac{8}{9}.$$

22. The number of different integers x that satisfy the equation $(x^2 - 5x + 5)^{(x^2 - 11x + 30)} = 1$ is

Sol. **(6)**

$\left(x^2 - 5x + 5\right)^{\left(x^2 + 11x + 30\right)} = 1$

Case- I

$x^2 - 11x + 30 = 0$
$x^2 - 6x - 5x + 30 = 0$
$x(x - 6) - 5(x - 6) = 0$
$(x - 6)(x - 5) = 0$
$x = 5, 6$

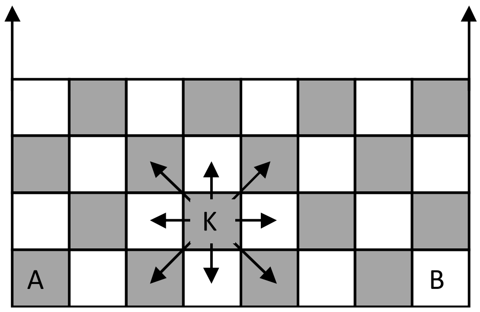
Case-II

$1 = x^2 - 5x + 5$
$x^2 - 5x + 4 = 0$
$x^2 - 4x - x + 4 = 0$
$x(x - 4) - 1(x - 4) = 0$
$(x - 4)(x - 1) = 0$
$x = 1, x = 4.$

Case - III

$x^2 - 5x + 5 = -1$ and $x^2 - 11x + 30$ = even
$x^2 - 5x + 6 = 0$
$x^2 - 3x - 2x + 6 = 0$
$x(x - 3) - 2(x - 3) = 0$
$(x - 3)(x - 2) = 0$
x = 2, 3 at x = 2 and 3
$x^2 - 11x + 30$ = even therefore x = 2, 3 are solutions. 6 answer.

23. In a single move a King K is allowed to move to any of the squares touching the square it is on, including diagonals, as indicated in the figure. The number of different paths using exactly seven moves to go from A to B is ___________.



Sol. **(127)**

Note : If King want to move from A to B in exact 7 moves then he can moves only the number marked in diagram and King can't move vertically up and can't move horizontally left.
Number of ways to move from A to (6, 7, 8, 9) in exact three moves.

A – 1 – 3 – 6

A – 1 – 3 – 7
A – 1 – 4 – 7
A – 2 – 4 – 7

A – 1 – 3 – 8
A – 1 – 4 – 8
A – 1 – 5 – 8
A – 2 – 5 – 8
A – 2 – 4 – 8

A – 2 – 5 – 9
A – 2 – 4 – 9
A – 1 – 4 – 9
A – 1 – 5 – 9

			6	10			
		3	7	11	14		
	1	4	8	12	15	17	
A	2	5	9	13	16	18	B

Number of ways to move from A to (6, 7, 8, 9) in exact three moves.

$A \rightarrow 6$ 1 way

$A \rightarrow 7$ 3 ways

$A \rightarrow 8$ 5 ways

$A \rightarrow 9$ 4 ways

Similarly number of ways to move toward B from (10, 11, 12, 13) in exact 3 moves.

$10 \rightarrow B$ 1 way

$11 \rightarrow B$ 3 ways

$12 \rightarrow B$ 5 ways

$13 \rightarrow B$ 4 ways

Number of ways to move from (6, 7, 8, 9) to (10, 11, 12, 13)

6 < 10, 11 — 2 ways

7 < 10, 11, 12 — 3 ways

8 < 11, 12, 13 — 3 ways

9 < 12, 13 — 2 ways

So the to number of ways from A to B is divided in three parts.

I. A to (6, 7, 8, 9)

II. (6, 7, 8, 9, 10) $\rightarrow$ (10, 11, 12, 13), then

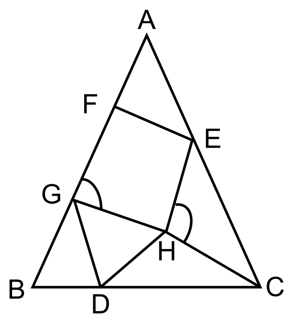
III. (10, 11, 12, 13) $\rightarrow$ B

1(1 + 3) + 3(1 + 3 + 5) + 5 (3 + 5 + 4) + 4 (5 + 4) = 4 + 27 + 60 + 36 = 127 ways.

Explanation :

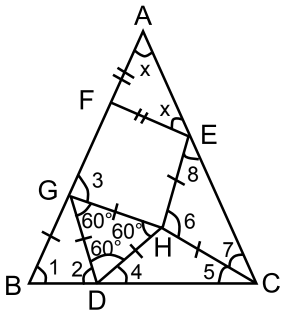
$$\underbrace{3}_{1\text{ to }7}\left(\underbrace{1}_{7\text{ to }10\text{ to }B}+\underbrace{3}_{7\text{ to }11\text{ to }B}+\underbrace{5}_{7\text{ to }12\text{ to}B}\right)$$

24. In ΔABC shows below, AB = AC, F is a point on AB and E a point on AC such that AF = EF, H is a point in the interior of ΔABC, D is a point on BC and G is a point on AB such that EH = CH = DH = GH = DG = BG. Also, $\angle CHE = \angle HGF$. The measure of $\angle BAC$ in degree is __________.



Sol. **(20)**

As AB = AC



$\therefore \quad \angle 1 = \frac{180 - x}{2} = 90 - \frac{x}{2}$

$\angle 1 = \angle 2 = 90 - \frac{x}{2}$

$\angle 3 + 60° = \angle 1 + \angle 2 = 180 - x$

$\angle 3 = 120 - x$

$\angle 4 = 180 - (\angle 2 + 60°) = 30 + \frac{x}{2}$

$\angle 5 = \angle 4 = 30 + \frac{x}{2}$

$\angle 6 = \angle 3 = 120 - x$

$\angle 7 = \angle 8 = \frac{180 - (120 - x)}{2} = 30 + \frac{x}{2}$

$\angle 7 + \angle 5 = \angle 1$

$30 + \frac{x}{2} + 30 + \frac{x}{2} = 90 - \frac{x}{2}$

$x = 20.$

25. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Then the value of $2x^3 + x^3y^3 + 2y^3$ is ____________.

Sol. **(89)**

$\frac{x^4y^2(x+y)}{x^3y^3(x^3+y^3)} = \frac{810}{945}$

$\frac{xy(x+y)}{x^3+y^3} = \frac{6}{7}$

$\frac{xy}{x^2+y^2-xy} = \frac{6}{7} \quad \Rightarrow \quad 6x^2 + 6y^2 - 13xy = 0$

$\Rightarrow \quad (3x - 2y)(2x - 3y) = 0$

$\frac{x}{y} = \frac{2}{3}$ or $\frac{y}{x} = \frac{2}{3}$

Let $x = \frac{2}{3}y$

$x^4 y^5 + y^4x^5 = 810$

$\left(\frac{2}{3}y\right)^4 y^5 + y^4\left(\frac{2}{3}y\right)^5 = 810$

$y^9 = \frac{3^9}{2^3} \quad \Rightarrow \quad y = \frac{3}{2^{1/3}} \quad \Rightarrow \quad y^3 = \frac{27}{2}$

$x = 2^{2/3} \quad \Rightarrow \quad x^3 = 4$

$\therefore \quad 2x^3 + 2y^2 + x^3y^3 = 2.4 + 2.\frac{27}{2} \;\; 4.\frac{27}{2} = 8 + 27 + 54 = 89$

26. The least odd prime factor of $2019^8 + 1$ is _________.

Sol. **(97)**

Let P be an odd prime which divides $2019^8 + 1$

So $2019^8 \equiv -1 \pmod{P}$

$\Rightarrow$ $2019^{16} \equiv 1 \pmod{P}$

Now by Euler's theorem

$2019^{P-1} \equiv 1 \pmod{P}$

So P – 1 should be divisible by 16

Where P is a prime

First two prime numbers which gives remainder 1 when divided by 16 is 17 and 97

Case- 1 P = 17

$2019^8 + 1 \equiv 13^8 + 1 \equiv 4^8 + 1 \equiv 16^4 + 1 \equiv 2$ (mode 17)

While

$2019^8 + 1 \equiv 79^8 + 1 = 18^8 + 1 = 324^4 + 1 \equiv 33^4 + 1 \equiv 1089^2 + 1 \equiv 22^2 + 1 \equiv 485 \equiv 0 \pmod{97}$

So the answer is 97.

27. Let a, b, c be positive integers each less than 50, such that $a^2 - b^2 = 100c$. The number of such triples (a, b, c) is

Sol. **(25)**

$a^2 - b^2 = 100\ c$

As $a^2 - b^2$ is a multiple of 100.

So it means the last 2 digit of a^2 and b^2 is same.

So (a, b) can be (49, 1) (48, 2) (47, 3)...........(26, 24)

So there are 24 such pairs

One more pair for (a, b) is (25, 15)

So total 25 pairs are possible.

28. The number of non-negative integers which can be written in the form $b_4 \cdot 3^4 + b_3 \cdot 3^3 + b_2 \cdot 3^2 + b_1 \cdot 3^1 + b_0 \cdot 3^0$, where $b_i \in \{-1, 0, 1\}$ for $0 \le i \le 4$ is ___________.

Sol. **(122)**

$b_4 \cdot 3^4 + b_3 \cdot 3^3 + b_2 \cdot 3^2 + b_1 \cdot 3^1 + b_0 \cdot 3^0$

Case - I : $b_4 = 1$ than we can take any value for b_3, b_2, b_1, b_0

so total number formed = 3^4

Case - II : $b_4 = 0$ and $b_3 = 1$ than we can take any value for b_2, b_1, b_0

so total number formed = 3^3

Case - III : $b_4 = 0$, $b_3 = 0$ and $b_2 = 1$ than we can take any value for b_1, b_0

so total number formed = 3^2

Case - IV : $b_4 = 0$, $b_3 = 0$, $b_2 = 0$ and $b_1 = 1$ than we can take any value for b_0

so total number formed = 3

Case - V : $b_4 = b_3 = b_2 = b_1 = 0$, than b_0 can take value 0, 1

so total number formed = 2

So total number formed = $3^4 + 3^3 + 3^2 + 3 + 2$

$= 81 + 27 + 9 + 3 + 2 = 122.$

29. $\{a_k\}$ is a sequence of integers, with $a_1 = -2$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m, n. Then the value of a_8 = ________.

Sol. **(12)**

$a_1 = -2$

$a_2 = a_{1+1} = a_1 + a_1 + 1 \cdot 1$

$= -2 - 2 + 1 = -3$

$a_4 = a_{2+2} = a_1 + a_2 + 2 \cdot 2$

$= -3 - 3 + 4$

$a_4 = -2$

$a_8 = a_{4+4} = a_4 + a_4 + 4 \times 4$

$= -2 - 2 + 16 = 12.$

30. The coefficient of x^{90} in $(1 + x + x^2 + x^3 + ... + x^{60})(1 + x + x^2 + ... + x^{120})$ is equal to _______.

Sol. **(61)**

The coefficient of x^{90} in $(1 + x + x^2 + x^3 + ... + x^{60})(1 + x + x^2 + ... + x^{120})$ is obtained by when

$(1 \times x^{90}) + (x \times x^{89}) + (x^2 \times x^{88}) + ... + (x^{60} \times x^{30})$

So, there are 61 terms in which the power x is 90 and there coefficient 1 so the coefficient of x^{90} is 61.

Alternate :

$(1 + x + x^2 + x.......... + x^{60})(1 + x + x^2 + ... + x^{120})$

$= \left(\frac{1-x^{61}}{1-x}\right)\left(\frac{1-x^{121}}{1-x}\right)$

Coefficient of x^{90} in $\left(\frac{1-x^{61}}{1-x}\right)\left(\frac{1-x^{121}}{1-x}\right)$

$= (1 - x^{61})(1 - x^{121})(1 - x)^{-2}$

= coefficient of x^{90} in $(1 - x)^{-2}$ – coefficient of x^{29} in $(1 - x)^{-2}$

$= {}^{90+2-1}C_{2-1} - {}^{29+2-1}C_{2-1}$

$= {}^{91}C_1 - {}^{30}C_1 = 91 - 30 = 61.$