KAPREKAR CONTEST - FINAL - SUB JUNIOR

Classes VII & VIII

AMTI - Saturday, 3rd November_2018.

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. A lucky year is one in which at least one date, when written in the form day / month / year, has the following property. The product of the month times the day equals the last two digits of the year. For example, 1956 is a lucky year because it has the date 7 / 8 / 56 where 7 × 8 = 56, but 1962 is not a lucky year as 62 = 62 × 1 or 31 × 2, where 31/2/1962 is not a valid date. From 1900 to 2018 how many years are not lucky (not including 1900 and 2018) ? Given proper explanation for your answer.

Sol. Month × Date = {1, 2, 3, ..., 99}

Month ⇒ {1, 2, 3,........., 12}

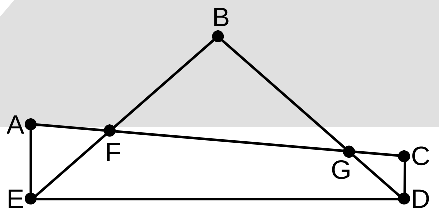
Date ⇒ {1, 2, 3,........., 31}

Years which are not lucky

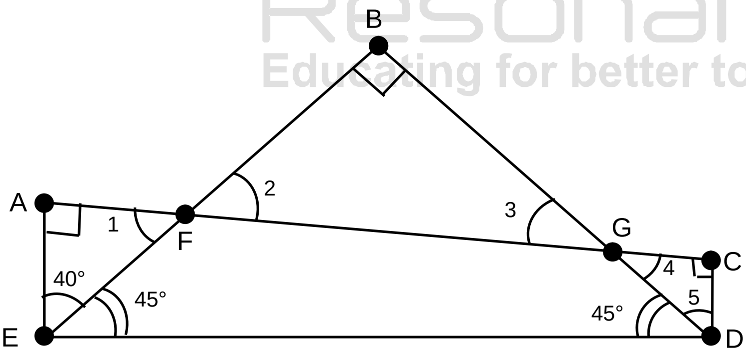
1937, 1941, 1943, 1947, 1953, 1958, 1959, 1961, 1962, 1967, 1971, 1973, 1974, 1979, 1982, 1983, 1986, 1989, 1994, 1997, 2000.

Total = 21 years.

2. In the figure given, ∠A, ∠B and ∠C are right angles. If and ∠AEB = 40° and ∠BED = ∠BDE, then find ∠CDE.



Sol.



∠BED = ∠BDE = 45° {∠B = 90°}

∠1 = 180° – (90° + 40°) = 50°, ∠1 = ∠2 = 50°

$\angle 3 = 180 - (90° + \angle 2)$
$= 180° - 90° - 50° = 40°$
$\angle 3 = \angle 4 = 40°$
$\angle 5 = 180° - (90° + \angle 4)$
$= 180° - (90° + 40°) = 50°.$
$\angle CDE = \angle 5 + 45° = 50° + 45° = 95°.$

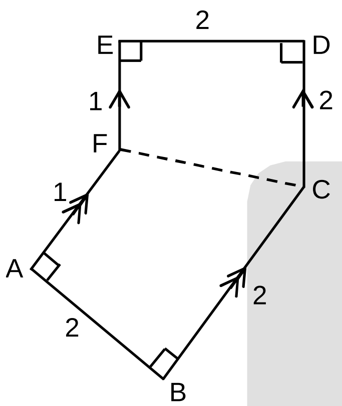
3.

(a) ABCDEF is a hexagon in which AB = BC = CD = DE = 2 and EF = FA = 1. Its interior angle C is between 90° and 180° and F is greater than 180°. The rest of the angles are 90° each. What is its area ?

(b) A convex polygon with 'n' sides has all angles equal to 150°, except one angle. List all possible values of n.

Sol.

(a)



Clearly FE || DC

$\therefore$ FEDC is trapezium

ar (FEDC) = $\frac{1}{2}$ (1 + 2) 2 = 3 ....(1)

Clearly FABC is trapezium

$\therefore$ ar (FABC) = $\frac{1}{2}$ (1 + 2) 2 = 3 ....(2)

Adding (1) and (2)

ar (hexagon) = 3 + 3 = 6.

(b) (n – 1) 150° + x = (n – 2)180

150n – 150 + x = 180n – 360

210 + x = 30n

$n = \frac{210}{30} + \frac{x}{30}$

$n = 7 + \frac{x}{30}$

n is natural number so x should be multiple of 30°. But as polygon is convex so x < 180°.

So x = 30°, 60°, 90°, 120°. Accordingly n can take 8, 9, 10, 11.

4. a, b, c are distinct non-zero reals such that $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c}$. Find all possible values of $a^3 + b^3 + c^3$.

Sol. $\frac{1+a^3}{a} = \frac{1+b^3}{b}$

$b + ba^3 = a + ab^3$

$b - a = ab^3 - ba^3$

$(b - a) = ab (b^2 - a^2)$

$(b - a) = ab (b - a) (b + a)$

$ab (b + a) = 1$ ....(i)

$$\frac{1+b^3}{b} = \frac{1+c^3}{c}$$

$c + cb^3 = b + bc^3$

$c - b = bc^3 - cb^3$

$(c - b) = bc(c^2 - b^2)$

$bc(c + b) = 1$ ....(ii)

From equation (i) and (ii)

$$\frac{ab(b+a)}{bc(c+b)} = 1$$

$a(b + a) - c(c + b) = 0$

$ab + a^2 - c^2 - bc = 0$

$a^2 - c^2 + b(a - c) = 0$

$(a - c)(a + b + c) = 0$

$\Rightarrow a + b + c = 0$

$\Rightarrow a + b = -c$ ....(iii)

From equation (iii) in equation (i)

$ab(-c) = 1$

$abc = -1$

Now, $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$

$= 0 + 3(-1) = -3.$

Alternate

Let $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c} = k$

$a^3 - ak + 1 = 0,\ b^3 - bk + 1 = 0,\ c^3 - ck + 1 = 0$

Let assume a cubic equation

$x^3 - xk + 1 = 0$

Clearly its roots are a, b, c

$\therefore$ $a + b + c = 0$

$abc = -1.$

$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$

$= (0)(a^2 + b^2 + c^2 - ab - bc - ca) + 3(-1) = 0 - 3 = -3.$

5. Find the smallest positive integer such that it has exactly 100 different positive integer divisors including 1 and the number itself.

Sol. $100 = 2^2 \times 5^2 = 2 \times 2 \times 5 \times 5 = (1 + 1)(1 + 1)(4 + 1)(4 + 1)$

$N = P_1^4 P_2^4 P_3^1 P_4^1$

$N = 2^4 3^4 5^1 7^1 = 45, 360.$

6.

(a) What is the sum of the digits of the smallest positive integer which is divisible by 99 and has all of its digits equal to 2 ?

(b) When 270 is divided by the odd number n, the quotient is a prime number and the remainder is 0. What is n ?

Sol.

(a) As number is divisible 99

$\therefore$ it should be divisible by 9 and 11.

for divisibility by 9 the sum of digit is divisible by 9. As the number contains only 2 as digit so the sum of digit should be 18, 36,....... But the number should be divisible by 11 so we can't take sum as 18. So we take sum of digit is 36. So required number = 2222........... up to 18 times. Which is divisible by 11 and 9 i.e., 99.

Sum of the digits = 36.

(b) $\frac{270}{n}$ = Prime where n is odd

or $\frac{270}{\text{prime}} = n$ so it is possible only when prime is even so prime = 2.

$\therefore$ $\frac{270}{2} = 135 = n.$

7. Consider the sums

$$A = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + ...+ \frac{1}{99\cdot 100} \text{ and } B = \frac{1}{51\cdot 100} + \frac{1}{52\cdot 99} + ...+ \frac{1}{100\cdot 51}.$$

Express $\frac{A}{B}$ as an irreducible fraction.

Sol. $A = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + ...+ \frac{1}{99\cdot 100}$

$= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{99} - \frac{1}{100}$

$= \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{100}\right] - 2\left[\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{100}\right]$

$= \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{100}\right] - \left[\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{50}\right]$

$A = \left[\frac{1}{51} + \frac{1}{52} + ... + \frac{1}{100}\right]$

$B = \frac{1}{51\cdot 100} + \frac{1}{52\cdot 99} + ...+ \frac{1}{100\cdot 51}$

$= \frac{151}{151}\left[\frac{1}{51\cdot 100} + \frac{1}{52.99} + ... + \frac{1}{100.51}\right]$

$= \frac{1}{151}\left[\frac{151}{51\cdot 100} + \frac{151}{52.99} + ... + \frac{151}{100.51}\right]$

$= \frac{1}{151}\left[\frac{51+100}{51\cdot 100} + \frac{52+99}{52.99} + ... + \frac{100+51}{100.51}\right]$

$= \frac{1}{151}\left[\frac{1}{100} + \frac{1}{51} + \frac{1}{99} + \frac{1}{52} + ... + \frac{1}{.51} + \frac{1}{100}\right]$

$= \frac{2}{151}\left[\frac{1}{51} + \frac{1}{52} + +........\frac{1}{100}\right]$

$$\frac{A}{B} = \frac{\left[\frac{1}{51} + \frac{1}{52} + +........\frac{1}{100}\right]}{\frac{2}{151}\left[\frac{1}{51} + \frac{1}{52} + +........\frac{1}{100}\right]} = \frac{151}{2}.$$

8. Let a, b, c be real numbers, not all of them are equal. Prove that if a + b + c = 0, then $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$.
Prove the converse, if $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 = ca + a^2$, then a + b + c = 0.

Sol.

I. $a^2 + ab + b^2 = (-b-c)^2 + (-b-c)\,b + b^2$ {as a + b + c = 0, a = – b – c}
$= (b^2 + c^2 + 2bc) - b^2 - bc + b^2$
$= b^2 + c^2 + bc$
Similarly $b^2 + c^2 + bc = c^2 + ca + a^2$
$\therefore a^2 + ab + b^2 = b^2 + c^2 + bc = c^2 + a^2 + ac.$

II. $a^2 + ab + b^2 = b^2 + bc + c^2$
$a^2 + ab - bc - c^2 = 0$
$a^2 - c^2 + b(a - c) = 0$
$(a - c)(a + c + b) = 0$
$a - c = 0$ or $a + b + c = 0$
as a – c = 0 is not possible as a, b, c are not equal.
$\therefore \quad a + b + c = 0.$

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