BHASKARA CONTEST - FINAL - JUNIOR

Classes IX & X

AMTI - Saturday, 3rd November_2018.

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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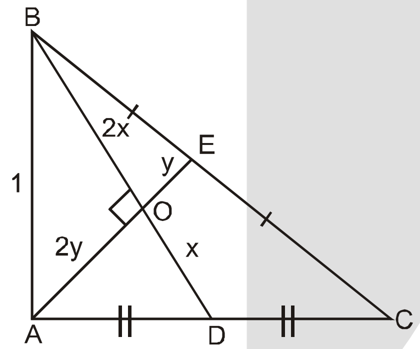
1. ABC is a right angled triangle with BC as hypotenuse. The medians drawn to BC and AC are perpendicular to each other. If AB has length 1 cm, find the area of triangle ABC.

Sol.



Let OD = x and OE = y

Hence BO = 2x and AO = 2y

In ΔAOB

$(2x)^2 + (2y)^2 = 1$

$4(x^2 + y^2) = 1$

$4x^2 = 1 - 4y^2$ ... (1)

In ΔBOE

$BE^2 = 4x^2 + y^2 = EC^2$ .... (2)

In ΔAOD

$AD^2 = 4y^2 + x^2 = DC^2$ ... (3)

Now in ΔABC

$AB^2 + AC^2 = BC^2$

$1^2 + [2(AD)]^2 = [2(BE)]^2$

$1 + 4 \times (4y^2 + x^2) = 4(4x^2 + y^2)$

From equation (2) & (3)

$1 + 16y^2 + 4x^2 = 16x^2 + 4y^2$

$1 + 12y^2 = 12x^2 = 3 \times 4x^2$

From equation (1)

$1 + 12y^2 = 3 \times (1 - 4y^2)$

$1 + 12y^2 = 3 - 12y^2$

$$24y^2 = 2$$

$$y^2 = \frac{1}{12}$$

$$y = \sqrt{\frac{1}{12}} \qquad \text{... (4)}$$

From equation (1)

$$4x^2 = 1 - 4 \times \frac{1}{12} = \frac{2}{3}$$

$$x^2 = \frac{1}{6}$$

$$x = \sqrt{\frac{1}{6}}$$

Now from equation (3)

$$AD^2 = 4 \times \frac{1}{12} + \frac{1}{6}$$

$$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$AD = \sqrt{\frac{1}{2}}$$

And $AC = 2 \times AD = 2 \times \sqrt{\frac{1}{2}} = \sqrt{2}$

Area of ΔABC

$$\frac{1}{2} \times AB \times AC = \frac{1}{2} \times 1 \times \sqrt{2} = \frac{1}{\sqrt{2}} cm^2$$

2.

(a) Find the smallest positive integer such that it has exactly 100 different positive integer divisors including 1 and the number itself.

(b) A rectangle can be divided into 'n' equal squares. The same rectangle can also be divided into (n + 76) equal squares. Find n.

Sol.

(a) $100 = 2^2 \times 5^2 = 2 \times 2 \times 5 \times 5 = (1 + 1)(1 + 1)(4 + 1)(4 + 1)$

$N = P_1^4 P_2^4 P_3^1 P_4^1$

$N = 2^4 3^4 5^1 7^1$

= 45, 360.

(b) In case I we have x square in horizontal line and y squares in vertical line so n = xy (total number of square)

Similarly n + 76 = uv

The ratio of horizontal and vertical number of square in both cases will be same.

So, $\frac{x}{u} = \frac{y}{v} \Rightarrow xv = uy$

Denote HCF (x, u) = a

HCF (y, v) = b

then we can write x = ac

u = ad where HCF (c, d) = 1

Similarly y = bc

v = bd where HCF (c, d) = 1

So, $uv - xy = (d^2 - c^2)\,ab = 76$

$(d + c)(d - c)\,ab = 76 = 2^2 \times 19$

$\Rightarrow$ $d - c = 1,\ d + c = 19$

$ab = 4$

$d = 10$

$c = 9,\ ab = 4$

Hence $n = xy = c^2ab$

$= 9^2 \times 4 = 324.$

3. Prove that $1^n + 2^n + 3^n + ... + 15^n$ is divisible by 480 for all odd $n \geq 5$.

Sol. $1^n + 2^n + 3^n + ... + 15^n$

$(1^n + 15^n) + (2^n + 14^n) + (3^n + 13^n) + ... + (7^n + 9^n) + 8^n$

$$\underbrace{\left[(1^n+15^n) + (3^n+13^n) + (5^n+11^n) + (7^n+9^n)\right]}_{A} + \underbrace{\left[(2^n+14^n) + (4^n+12^n) + (6^n+10^n) + 8^n\right]}_{B}$$

In expression B

$\because n \geq 5.$

Hence all numbers are multiple of 32.

In expression A

$[1^n + (16 - 1)^n] + [3^n + (16 - 3)^n] + [5^n + (16 - 5)^n] + [7^n + (16 - 7)^n]$

$\because$ n is odd.

$$\left[1^n + \left\{\underbrace{16^n - {}^nC_1\,16^{n-1} + ... + 16\,{}^nC_{n-1}(-1)^{n-1} + (-1)^n}_{\text{number of terms even}}\right\}\right]$$

$$+ \left[3^n + \left\{\underbrace{16^n + {}^nC_1\,16^{n-1}(-3) + ... + {}^nC_{n-1} \times 16\,(-3)^{n-1} + (-3)^n}_{\text{number of terms even}}\right\}\right] + [....................]$$

$\because$ n is odd then $(-1)^n = -1$

$$\left[\underbrace{16^n - {}^nC_1\,16^{n-1} + ... + \underset{\uparrow\ \text{odd}}{n} \times 16}_{\text{number of terms odd}}\right] + \left[\underbrace{16^n + {}^nC_1\,16^{n-1}(-3)\ ....\ n \times 16 \times (-3)^{n-1}}_{\text{number of terms odd}}\right] + ...$$

Taking 16 as a common, sum of remaining numbers are odd because n is odd.

$16k_1 + 16k_2 + 16k_3 + 16k_4$ {where k_1, k_2, k_3, k_4 are odd}

$$16\,\underbrace{(k_1 + k_2 + k_3 + k_4)}_{\text{even}}$$

Means A is multiple of 32.

So, A + B is also multiple of 32.

Now, $(1^n + 2^n) + (4^n + 5^n) + (7^n + 8^n) + ... (13^n + 14^n) + \{3^n + 6^n + ... 15^n\}$

$(x^n + y^n)$ is always divisible by $(x + y)$ when n is odd means all factors is divisible by 3.

Now, $(1^n + 4^n) + (2^n + 3^n) + (6^n + 9^n) + (7^n + 8^n) + (11^n + 14^n) + (12^n + 13^n) + \{5^n + 10^n + 15^n\}$

$(x^n + y^n)$ is always divisible by $(x + y)$ when n is odd.

Means all factors is divisible by 5.

Hence given expression is divisible by $32 \times 3 \times 5 = 480$.

4. Is it possible to have 19 lines in a plane such that (1) no three lines have a common point and (2) they have exactly 95 points of intersection. Validate.

Sol. Let assume that r lines out of 19 lines are parallel to each other.

$^{19}C_2 - ^{r}C_2 = 95$

$\frac{19 \times 18}{2} - \frac{r(r-1)}{2} = 95$

19 × 18 – r (r – 1) = 190

r(r – 1) = 19 × 18 – 190

= 19(18 – 10)

r (r – 1) = 19 × 8

Which does not give a natural value of r. So such condition is not possible

5. In a trapezium ABCD with AB parallel to CD, the diagonals intersect at P. The area of ΔABP is 72 cm^2 area of ΔCDP is 50 cm^2. Find the area of the trapezium.

Sol. ar ΔABC = ar ΔABD

(Δ's have same base and between some parallel)

Subtract ar ΔABP from both side, we get

ar ΔADP = Ar ΔBPC = x

And we also know that

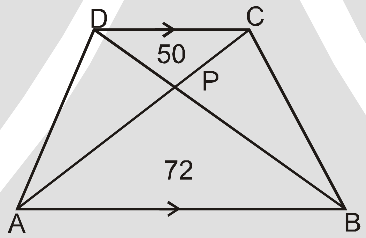
ar ΔABP × ar ΔDPC = ar ΔAPD × ar ΔBPC

$72 \times 50 = x^2$

x = 60

So area of trapezium ABCD

= 60 + 60 + 72 + 50 = 242



6. Let a < b < c be three positive integers. Prove that among any 2c consecutive positive integers there exists three different numbers x, y, z such that abc divides xyz.

Sol. In solving this question, we use the fact.

"If we take n consecutive number then one of them should be divisible by n"

If we take 2c consecutive number. Then there should be one no (X) among the which is divisible by c.

As a and b are less than c, so it must contain the number (ZY) which divisible by a and b.

So we can say XYZ in divisible by abc.

Let take an example a = 2, b = 3, c = 5.

Now we take 2c (i.e., 2 × 5 = 10) consecutive number.

Let they 23, 24, 25, ..., 32

We see that

X = 24 is divisible by 2 = a

Y = 27 is divisible by 3 = b

Z = 25 is divisible by 5 = c

$\therefore$ XYZ is divisible by abc.

7.

(a) Let m, n be positive integers. If $m^3 + n^3$ is the square of an integer, then prove that (m + n) is not a product of two different prime numbers.

(b) a, b, c are real numbers such that, ab + bc + ca = – 1. Prove $a^2 + 5b^2 + 8c^2 \geq 4$.

Sol.

(a) Let us assume that m + n = pq, where p and q are distinct prime.

Since $m^3 + n^3 = (m + n)(m^2 - mn + n^2)$

$= (m + n)[(m + n)^2 - 3mn]$

is a square so $m^3 + n^3 = (m + n)[(m + n)^2 - 3mn]$

must divisible by pq.

$\Rightarrow$ 3mn must divisible by p and q.

Since $p \neq q$ let $q \neq 3$ so $q \mid m$ but $q \mid m + n$ so $q \mid n$

m = qx, n = qy

$\Rightarrow$ p = 3.

so m + n = pq = 3q

$\Rightarrow$ qx + qy = 3q

$\Rightarrow$ m = 2q, n = q

or m = q, n = 2q

$m^3 + n^3 = 9q^3$ is not a square of an integer

which is a contradiction

Hence m + n is not a product of two different prime.

(b) $(2a + 5b + 2c)^2 \geq 0$, $(a + 6c)^2 \geq 0$

By adding both equation

$5a^2 + 25b^2 + 40c^2 + 20ab + 20bc + 20ac \geq 0$

$\Rightarrow$ $5(a^2 + 5b^2 + 8c^2) \geq -20(ab + bc + ca)$

$\Rightarrow$ $5(a^2 + 5b^2 + 8c^2) \geq -20(-1)$

$\Rightarrow$ $a^2 + 5b^2 + 8c^2 \geq 4$.

Alternate :

$(x + 2y + 2z)^2 \geq 0$

$x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 8yz \geq 0$ ....(i)

and

$(y - 2z)^2 \geq 0$

$y^2 - 4yz + 4z^2 \geq 0$ ....(ii)

By adding (i) and (ii)

$x^2 + 5y^2 + 8z^2 + 4(xy + yz + xz) \geq 0$

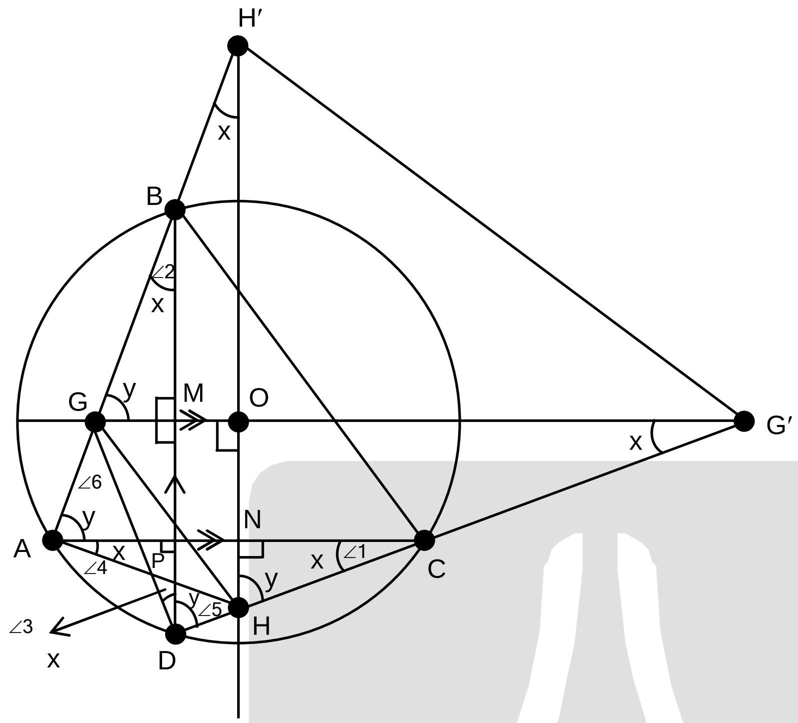
$x^2 + 5y^2 + 8z^2 - 4 \geq 0$

$x^2 + 5y^2 + 8z^2 \geq 4$

H.P.

8. ABCD is a quadrilateral in a circle whose diagonals intersect at right angles. Through O the centre of the circle, GOG′ and HOH′ are drawn parallel to AC, BD respectively, meeting AB, CD in G, H and DC, AB produced in G′, H′. Prove GH, G′H′ are parallel to BC and AD respectively.

Sol.



$\angle 1 = \angle 2 = x$ (Angle is same segment)

O is centre OM and ON are perpendicular bisector of BD and AC

$\Delta GDM \cong \Delta GBM$

$\angle 3 = \angle 2 = x$

Similarly, $\angle 4 = \angle 1 = x$

$\angle 5 = \angle 6 = y$ (Angle is same segment)

ADHG is cyclic quadrilateral ($\angle HAG = \angle HDG = x + y$) angle in same segment

$\Rightarrow$ $\angle GHC = \angle GAD = \angle BCG$ (Cyclic quadrilateral exterior angle prop.)

Hence HG || BC.

Quadrilateral HGH′G′ is cyclic quadrilateral

$\angle GH'G' = \angle DHG$ ....(i)

But $\angle DHG + \angle DAG = 180°$

$\angle DAG = 180° - \angle DHG$ ....(ii)

Now $\angle GH'G' + \angle DAG = \angle DHG + 180° - \angle DHG = 180°$ (Co interior angle)

Hence, AD || H′G′.

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