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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST - FINAL - JUNIOR
Classes IX & X
Saturday, 28th October_2017.

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

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1. **(a)** Find all prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are also primes.
(b) Determine real numbers x, y, z, u such that
$xyz + xy + yz + zx + x + y + z = 7$
$yzu + yz + zu + uy + y + z + u = 9$
$zux + zu + ux + xz + z + u + x = 9$
$uxy + ux + xy + yu + u + x + y = 9$

Sol. **(a)** Let P = 2
$4P^2 + 1 = 4(2)^2 + 1 = 17$
$6P^2 + 1 = 6(2)^2 + 1 = 25$ (not prime)
Let P = 3
$4(3)^2 + 1 = 37$
$6(3)^2 + 1 = 55$ not prime
P = 5
$4(5)^2 + 1 = 101$
$6(5)^2 + 1 = 151$
So $4P^2 + 1$ and $6P^2 + 1$ both are prime for P = 5
We know that every square no. is of the form 5 m, 5m+ 1 or 5m + 4
Let take prime P > 5
So P can be 5m + 1 or 5m + 4
Case – I
5m + 1
$4P^2 + 1 = 4(5m + 1)^2 + 1 = 20k + 5 = 5(4k + 1)$ (A multiple of 5)
Case – II
5m + 4
$6P^2 + 1 = 6(5m + 4)^2 + 1 = 30n + 25 = 5(6n + 5)$ (A multiple of 5)
So P = 5 is the only solution.

(b) $xyz + xy + yz + zx + x + y + z = 7$
$xy(z + 1) + y(z + 1) + x(z + 1) + (z + 1) = 8$
$(z + 1)(xy + y + x + 1) = 8$
$(z + 1)(x + 1)(y + 1) = 8$ … (1)
Similarly $(u + 1)(y + 1)(z + 1) = 10$ … (2)
$(x + 1)(z + 1)(u + 1) = 10$ … (3)
$(y + 1)(u + 1)(x + 1) = 10$ … (4)
Multiply
$(x + 1)^3 (y + 1)^3 (z + 1)^3 (u + 1)^3 = 8000$
$(x + 1)(y + 1)(z + 1)(u + 1) = 20$ … (5)

So, equation (5)/(1)

$$u+1=\frac{20}{8}\Rightarrow u+1=\frac{5}{2}\Rightarrow u=\frac{3}{2}$$

$$x + 1 = \frac{20}{10} \Rightarrow x + 1 = 2 \Rightarrow x = 1$$

y = 1

z = 1

2. If x, y, z, p, q, r are distinct real numbers such that

$$\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}=\frac{1}{p}$$

$$\frac{1}{x+q}+\frac{1}{y+q}+\frac{1}{z+q}=\frac{1}{q}$$

$$\frac{1}{x+r}+\frac{1}{y+r}+\frac{1}{z+r}=\frac{1}{r}$$

find the numerical value of $\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$.

Sol. $\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}=\frac{1}{p}$

Let $t = \frac{1}{p}$

then $\frac{1}{x+\frac{1}{t}}+\frac{1}{y+\frac{1}{t}}+\frac{1}{z+\frac{1}{t}} = t$

$$\frac{t}{tx+1}+\frac{t}{ty+1}+\frac{t}{tz+1}=t$$

$\Rightarrow$ (tx + 1) (ty + 1) + (tz + 1) (tx + 1) + (tz + 1) (ty + 1)
= (tx + 1) (ty + 1) (tz + 1)

Now, this cubic equation has roots $\frac{1}{p},\frac{1}{q},\frac{1}{r}$

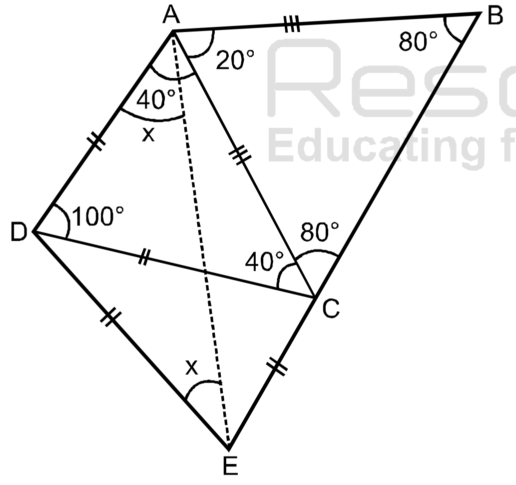
$$\Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=-\frac{(\text{coefficient of } t^2)}{\text{coefficient of } t^3}$$

Solving equation we get coefficient of $t^2 = 0$.

$$\Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0.$$

3. ADC and ABC are triangles such that AD = DC and CA = AB . If ∠CAB = 20° and ∠ADC = 100°, without using Trigonometry, prove that AB = BC + CD.

Sol.



Extend BC to E such that CE = CD.
Now, CED is equilateral triangle join AE. Let $\angle DAE = x$
then $x = \angle DEA$
$\angle AEC = 60 - x$
$\angle EAC = 40 - x$
$\angle EAB = 60 - x$
ΔABE is isosceles
AB = BE
= BC + CE
= BC + CD.

4. **(a)** a, b, c, d are positive real numbers such that abcd = 1.

Prove that $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d} \geq 4$

(b) In a scalene triangle ABC, $\angle BAC = 120°$. The bisectors of the angles A, B and C meet the opposite sides in P, Q and R respectively. Prove that the circle on QR as diameter passes through the point P.

Sol. **(a)** $\frac{1+ab}{1+a}=\frac{abcd+ab}{abcd+a}=\frac{bcd+b}{bcd+1}=1+\frac{b-1}{bcd+1}$

We have to prove,

$$\sum\frac{b-1}{bcd+1}\geq 0$$

Now

$$\sum\frac{(b-1)^2}{(bcd+1)(b-1)}\geq\frac{(a+b+c+d-4)^2}{\sum(bcd+1)(b-1)} \qquad \text{(i)}$$

by Titu's lemma (extended cauchy)

Now, let the expression $\sum(bcd+1)(b-1)$ be E.

$$E = \sum\left(\frac{1}{a}+1\right)(b-1)=\sum\left[\frac{b}{a}+b-\frac{1}{a}-1\right]$$

$$= \left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) + (a + b + c + d - 4)$$

$$= \left(\frac{b-1}{a}+\frac{c-1}{b}+\frac{d-1}{c}+\frac{a-1}{d}\right) + (a + b + c + d - 4)$$

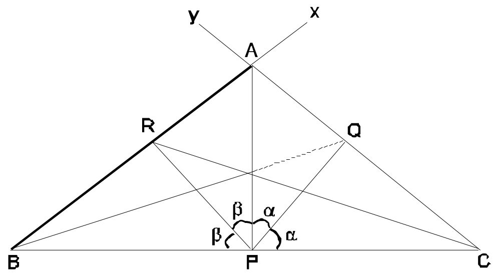
$a + b + c + d \geq 4$ by AM – GM inequality
$ab + bc + cd + da \geq 4$ by AM – GM inequality

Again, $\sum\frac{(b-1)^2}{a(b-1)}\geq\frac{(a+b+c+d-4)^2}{(ab+bc+cd+da-4)}\geq 0$

Hence, $E \geq 0$
$a + b + c + d \geq 4$ by AM-GM
Hence,

$$\sum\frac{(b-1)^2}{(bcd+1)(b-1)}\geq 0$$

(b)



Produce BA upto x and CA upto y

∠ BAP = ∠ CAP = 60° (Given)

∠ YAQ = ∠ PAQ = 60°

So AQ is bisector of exterior ∠ A of Δ ABP

BQ is bisector of interior ∠ B of Δ ABP

So Q is excentre of ΔABP

So PQ bisect ∠ APC

Let ∠ APQ = ∠ CPQ = α (say)

Similarly ∠ BPA = ∠ QPA = β …………

∠ BPA = ∠ CPA = 180°

$2\beta + 2\alpha = 180°$

$\alpha + \beta = 90°$

∠ QPR = 90°

Hence circle on QR as diameter passes through point P.

5. **(a)** Prove that $x^4 + 3x^3 + 6x^2 + 9x + 12$ cannot be expressed as a product of two polynomials of degree 2 with integer coefficients.

(b) 2n + 1 segments are marked on a line. Each of these segments intersects at least n other segments. Prove that one of these segments intersects all other segments.

Sol. **(a)** Let $x^4 + 3x^3 + 6x^2 + 9x + 12$

$= (x^2 + Ax + B)(x^2 + Cx + D)$

$= x^4 + Cx^3 + Dx^2 + Ax^3 + ACx^2 + ADx + Bx^2 + BCx + BD$

$= x^4 + (A + C)x^3 + (D + AC + B)x^2 + (AD + BC)x + BD$

Now by comparing coefficient

A + C = 3

B + D + AC = 6

AD + BC = 9

BD = 12

Case – I : B = 1, D = 12

$\therefore$ A + C = 3

12A + C = 9 have no integer solution.

Case – II : B = – 1, D = – 12

C + 12 A = – 9

C + A = 3 have no integer solution.

Case – III : B = 2, D = 6

2C + 6A = 9

C + A = 3 have no integer solution.

Case – IV : B = – 2, D = – 6

2C + 6A = – 9

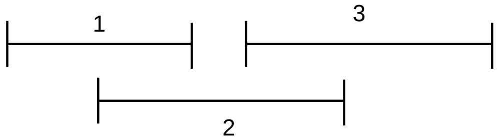
A + C = 3 have no integer solution.

So, $x^4 + 3x^3 + 6x^2 + 9x + 12$ cannot be expressed as a product of two polynomial of degree 2 with integer coefficient.

(b) The question is to be done by induction. Let's take k = 1

⇒ There are three segments such that all segments intersect at least one segment.

Only possibility is as follows :



Here 2 intersect both (1) & (3).
Hence, statement is true for k = 1
Note: This is the worst case in which every segment is intersecting k segments. If we can prove in this one, then it can be proved otherwise as well.

Let if be true for k = n – 1, it implies that 2 (n – 2) segments intersect (n – 1) segments and 1 segment intersect all other.
Now, we add 2 segments such that all the segments intersect with n segments. It means one of these segments will intersect with (n – 1) segments and other with another (n – 1) segments.
This way 2(n – 2) segments intersect with n segments.
Now, these two segments have (n – 1) intersection.
They have to intersect with the segment intersecting all other to satisfy. Hence proved by PMI.

6. If a, b, c, d are positive real numbers such that $a^2 + b^2 = c^2 + d^2$ and $a^2 + d^2 - ad = b^2 + c^2 + be$, find the value $\frac{ab+cd}{ad+bc}$.

Sol. $a^2 + b^2 = c^2 + d^2$

$(a + b)^2 – (c – d)^2 = 2 (ab + cd)$ …………(1)

$(c + d)^2 – (a – b)^2 = 2(ab + cd)$ …………(2)

(1) × (2)

$4(ab + cd)^2 = (a + b + c – d) (a + b – c + d)$
$(c + d + a – b) (c + d – a + b)$ …………(3)

Also $a^2 + d^2 – ad = b^2 + c^2 + bc$

$(a + d)^2 – (b – c)^2 = 3 (ad + bc)$ …………(4)

$(b + c)^2 – (a – d)^2 = (ad + bc)$ …………(5)

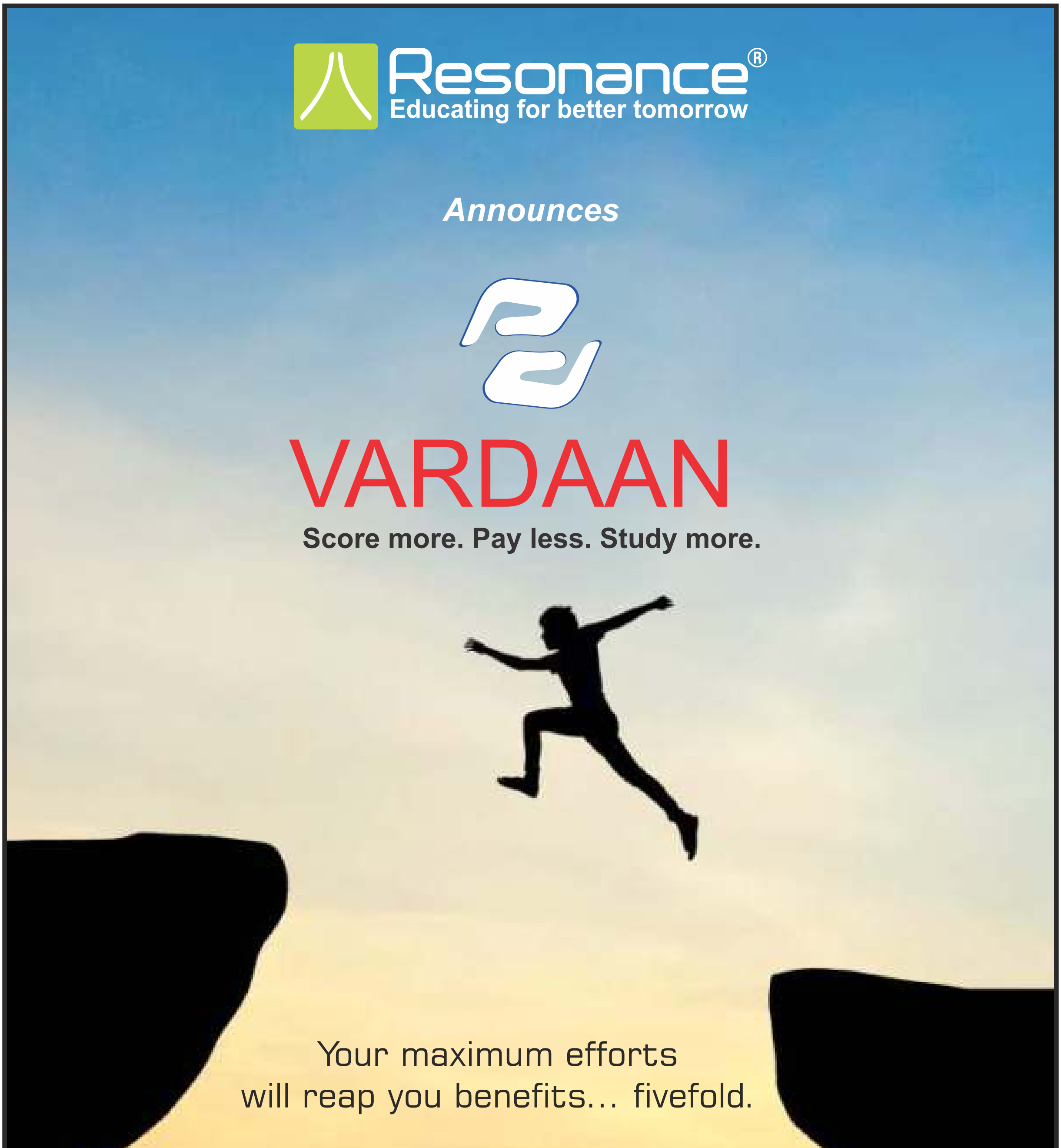
(4) × (5)

$3(ad + bc)^2 = (a + d + b – c) (a + d – b + c)$
$(b + c + a – d) (b + c – a + d)$ …………(6)

RHS of (3) and (6) is equal hence

$3(ad + bc)^2 = 4(ab + cd)^2$

$$\frac{3}{4} = \left(\frac{ab+cd}{ad+bc}\right)^2 \Rightarrow \frac{ab+cd}{ad+bc} = \frac{\sqrt{3}}{2}.$$



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for VIII IX X

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Note: Valid for few study centers