PAPER (पेपर)- 2

PAPER-2 : INSTRUCTIONS TO CANDIDATES

- **This question paper has three (03) parts: PART-I: Physics, PART-II: Chemistry and PART-III: Mathematics.**
- **Each part has total of eighteen (18) questions divided into three (03) Sections (Section-1, Section-2 and Section-3).**
- **Total number of questions in Paper-2 : Fifty four (54).**
- **Paper-2 Maximum Marks : One Hundred Eighty (180).**

Instructions for Section-1 : Questions and Marking Scheme

SECTION-1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR options** for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks :	**+4** If only (all) the correct option(s) is (are) chosen.
Partial Marks :	**+3** If all the four options are correct but ONLY three options are chosen.
Partial Marks :	**+2** If three or more options are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks :	**+1** If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks :	**0** If none of the options is chosen (i.e. the question is unanswered).
Negative Marks :	**–2** In all other cases.

- **For Example :** If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in –2 marks.

Answering Section-1 Questions :

- To select the option(s), **using the mouse click** on the corresponding button(s) of the option(s).
- To deselect chosen option(s), click on the button(s) of the chosen option(s) again or click on the **Clear Response** button to clear all the chosen options.
- To change the option(s) of a previously answered question, if required, first click on the **Clear Response** button to clear all the chosen options and then select the new option(s).
- To mark a question ONLY for review (i.e. without answering it), click on the **Mark for Review & Next button**.
- To mark a question for review (after answering it), click on **Mark for Review & Next button** – answered question which is also marked for review will be evaluated.
- To save the answer, click on the **Save & Next** button – the answered question will be evaluated.

Instructions for Section-2 : Questions and Marking Scheme

SECTION-2 (Maximum Marks : 24)

- This section contains **EIGHT (08)** questions. The answer to each question is **NUMERICAL VALUE**.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 6.25, 7.00, –0.33, –0.30,,30.27, –127.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : **+3** If ONLY the correct numerical value is entered as answer.

Zero Marks : **0** In all other cases.

Answering Section-2 Questions :

- Using the attached computer mouse, click on numbers (and/or symbols) on the **on-screen virtual numeric keypad** to enter the numerical value as answer in the space provided for answer.
- To change the answer, if required, first click on the **Clear Response** button to clear the entered answer and then enter the new numerical value.
- To mark a question ONLY for review (i.e. answering it), click on **Mark for Review & Next button** – the answered question which is also marked for review will be evaluated.
- To mark a question for review (after answering it), click **Mark for Review & Next button** – the answered question which is also marked for review will be evaluated.
- To save the answer, click on the **Save & Next button** – the answered question will be evaluated.

Instructions for Section-3 : Questions and Marking Scheme

SECTION-3 (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **TWO (02)** matching lists ; **LIST-I** and **LIST-II**.
- **FOUR options** are given representing matching of elements from LIST-I and LIST-II. ONLY ONE of these four each question, choose the option, choose the option corresponding to the correct matching.
- For each question, mark will be awarded according to the following marking scheme :

Full Marks : **+3** If ONLY the option corresponding to the correct matching is chosen.

Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : **–1** In all other cases.

Answering Section-3 Questions :

- To select an option, using the mouse click on the corresponding button of the option.
- To deselect the chosen answer, click on the button of the chosen option again or click on the **Clear Response button**.
- To change the chosen answer, click on the button of another option.
- To mark a question ONLY for review (i.e. without answering it), click on **Mark for Review & Next button**.
- To mark a question for review (after answering it), click on **Mark for Review & Next button** – the answered which is also marked for review will be evaluated.

To save the answer, click on the **Save & Next button** – the answered question will be evaluated.

PART : III MATHEMATICS

SECTION 1 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	**: +4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: **0**	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	**: –2**	In all other cases.

For Example: If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only all the three correct options will result in +4 marks. Selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option), without selecting any incorrect option (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in –2 marks.

1. For any positive integer n, define $f_n : (0, \infty) \to R$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

Then, which of the following statement(s) is (are) TRUE ?

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1+f_j'(0))\sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

Ans. **(D)**

Sol. $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$

$f_n(x) = \tan^{-1}(x+n) - \tan^{-1}(x) \quad \Rightarrow \quad f'_n(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

$f_n(0) = \tan^{-1}(n) \quad \Rightarrow \quad \tan^2(\tan^{-1}n) = n^2$

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = \sum_{j=1}^{5} j^2 = \frac{5.6.11}{6} = 55$ (since 0 is not in domain so A and B are wrong)

(B) $f'_n(0) = \frac{1}{1+n^2} - 1 \quad \Rightarrow \quad 1 + f_n(0) = \frac{1}{1+n^2}$

$\sec^2(f_n(0)) = \sec^2(\tan^{-1}(n)) = 1 + n^2$.

Hence $(1 + f'_n(0)).\sec^2(f_n(0)) = \left(\frac{1}{1+n^2}\right)(1+n^2) = 1$

so $\sum_{i=1}^{10}(1+f_i'(0))\sec^2(f_i(0)) = \sum_{i=1}^{10} 1 = 10$

$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \tan^{-1}\left(\frac{n}{1+x(n+x)}\right) = 0$

$\lim_{x\to\infty} \tan(f_n(x)) = 0 \quad \& \quad \lim_{x\to\infty} \sec^2(f_n(x)) = 1$

2. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE

(A) The point (–2, 7) lies in E_1

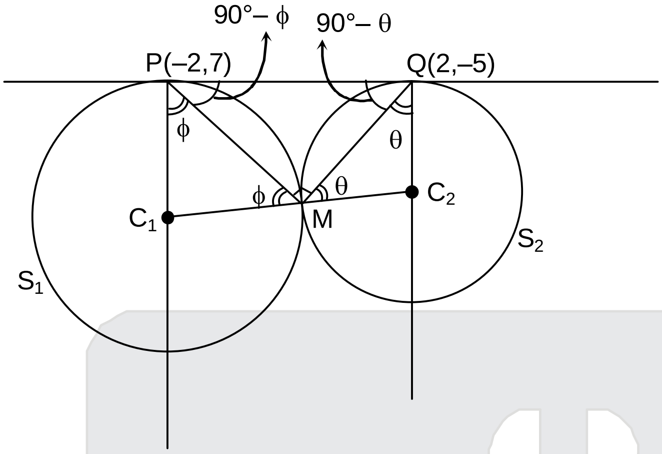
(B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2

(D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

Ans. **(BD)**

Sol.



Let C_1 and C_2 be the centre of circle S_1 and S_2 respectively

Let $\angle C_2QM = \angle C_2MQ = \theta \Rightarrow \angle QC_2M = \pi - 2\theta$

Let $\angle C_1PM = \angle C_1MP = \phi \Rightarrow \angle PC_1M = \pi - 2\phi$

Now $\angle QC_2M + \angle PC_1M = \pi \Rightarrow \pi - 2\theta + \pi - 2\phi = \pi \Rightarrow \theta + \phi = \pi/2$

Now $\angle QMP = \pi - \angle QMC_2 - \angle PMC_1 = \pi - (\theta + \phi) = \pi - \pi/2 = \pi/2$

hence locus equation of variable point M is $(x + 2)(x - 2) + (y - 7)(y + 5) = 0$

but locus of M does not contains point P and Q because P is included when radius of S_1 is zero and circle S_2 becomes straight line which is impossible. Q is included when radius of S_2 is zero and circle S_1 becomes straight line which is also impossible.

so set E_1 does not contain point P(–2, 7) and Q(2, – 5)

Locus of mid-points of chords passing through (1, 1) is $h + K - (1 + k) = h^2 + k^2 - 2K$

$\Rightarrow h^2 + K^2 - 2K - h + 1 = 0 \Rightarrow x^2 + y^2 - x - 2y + 1 = 0$

Now equation of line passing through P(–2, 7) and R(1, 1) is $\frac{y-1}{x-1} = \frac{6}{-3} \Rightarrow y + 2x - 3 = 0$

Let centre of $x^2 + y^2 - 2y - 39 = 0$ is C_3 (0, 1) $\Rightarrow$ centre of locus of M is C_3 (0, 1)

Now foot of $C_3(0, 1)$ on line $y + 2x - 3 = 0$ is $\left(\frac{4}{5}, \frac{7}{5}\right)$. which is mid-point of chord PR of circle $x^2 + y^2 - 2y - 39 = 0$

But if P is not the part of locus of M then PQ is not the chord of locus of M.

So point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does not lies in set E_2

3. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_2 \in R$ and the system of equation (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Ans. **(AD)**

Sol. $\Delta = 0$ so for at least one solutions $\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow b_1 + 7b_2 = 13b_3$ ....(1)

option (A) $\Delta \neq 0$ $\Rightarrow$ unique solution $\Rightarrow$ option (A) is correct

option (D) $\Delta \neq 0$ $\Rightarrow$ unique solution $\Rightarrow$ option (D) is correct

option (C) $\Delta = 0$ $\Rightarrow$ equations are $x - 2y + 5z = -b_1$

$$x - 2y + 5z = \frac{b_2}{2}$$

$$x - 2y + 5z = b_3$$

There planes are parallel so they must be coincident

$\Rightarrow$ $-b_1 = \frac{b_2}{2} = b_3$

All b_1, b_2, b_3 obtained from equation (1) may not satisfy this relation so option (C) is wrong.

option (B) $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 0$. Also $\Delta_1 = 0$

For infinite solutions, Δ_2 and Δ_3 must be 0

$\Rightarrow$ $\begin{vmatrix} 1 & b_1 & 1 \\ 5 & b_2 & 2 \\ 2 & b_3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $-b_1 - b_2 + 3b_3 = 0$ which does not satisfy (1) for all b_1, b_2, b_3 so option(B) is wrong

4. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE ?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

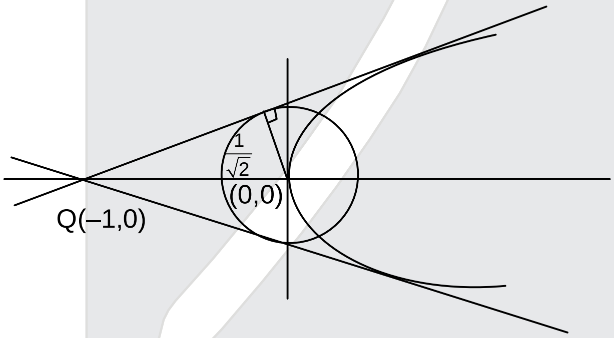
(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

Ans. (AC)

Sol.



Let equation of common tangent is $y = mx + \frac{1}{m}$

$\therefore \quad \left|\frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

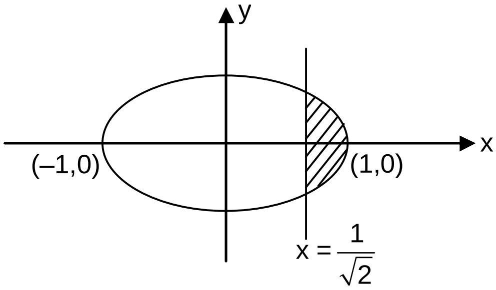
$\Rightarrow \quad m = \pm 1$

Equation of common tangents are $y = x + 1$ & $y = -x - 1$

point Q is (–1, 0)

$\therefore \quad$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $\quad e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$ $\quad$ & $\quad$ LR $= \frac{2b^2}{a} = 1$

(C) 

Area $2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}\,dx$

$$= \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4}-\left(\frac{1}{4}+\frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8}-\frac{1}{4}\right) = \frac{\pi-2}{4\sqrt{2}}$$

5. Let s, t, r be non-zero complex numbers and L be the set of solutions z = x + iy (x, y ∈ R, i = $\sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then , which of the following statement(s) is (are) TRUE ?

(A) If L has exactly one element, then |s| ≠ |t|

(B) If |s| = |t|, then L has infinitely many elements

(C) The number of elements in L ∩ {z : |z – 1 + i| = 5} is at most 2

(D) If L has more than one element, then L has infinitely many elements

Ans. **(ACD)**

Sol. $sz + t\bar{z} + r = 0$, $\bar{z} = x - iy$

$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0$

(1) + (2)

$(t + \bar{s})\bar{z} + (s + \bar{t})z + (r + \bar{r}) = 0$

$(t - \bar{s})\bar{z} + (s - \bar{t})z + (r - \bar{r}) = 0$

For unique solution

$\frac{t + \bar{s}}{t - \bar{s}} \neq \frac{s + \bar{t}}{s - \bar{t}}$

On solving the above equation we get

$|t| \neq |s|$

∴ option (A) is correct

Lines overlap if

$\frac{t + \bar{s}}{t - \bar{s}} = \frac{\bar{t} + s}{s - \bar{t}} = \frac{r + \bar{r}}{r - \bar{r}}$

$|t| = |s|$ $\quad\quad$ $\bar{t}r - \bar{t}\bar{r} + sr - s\bar{r} = sr + s\bar{r} - \bar{t}r - \bar{t}\bar{r}$

$2\bar{t}r = 2s\bar{r}$

$\bar{t}r = s\bar{r}$

$\therefore |\bar{t}||r| = |s||r|$

$\therefore |t| = |s|$

∴ If | t | = | s |, lines will be parallel for sure but it may not be coincident

For option (C) if element of set L represent line, then this line and given circle can have maximum two common points so option (C) is correct

6. Let f : (0, π) → R be a twice differentiable function such that $\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$ for all $x \in (0, \pi)$. If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

Ans. **(BCD)**

Sol. $\lim\limits_{t\to x} \dfrac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$

$\dfrac{f(x)\cos x - f'(x)\sin x}{\sin^2 x} = 1$

$-d\left(\dfrac{f(x)}{\sin x}\right) = 1$

$\dfrac{f(x)}{\sin x} = -x + c \quad \because \quad f\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{12} \quad \Rightarrow \quad c = 0 \quad \Rightarrow \quad f(x) = -x \sin x$

(A) $f(x) + f''(x) = -2\cos x$

$f\left(\dfrac{\pi}{2}\right) + f''\left(\dfrac{\pi}{2}\right) = 0$

(B) $f\left(\dfrac{\pi}{4}\right) = \left(-\dfrac{\pi}{4\sqrt{2}}\right)$

(C) f(x) is continuous and differentiable and f(0) = f(x) = 0

Using Rolle's theorem $f'(c) = 0$ for some $x \in (0, \pi)$

(D) $g(x) = -x\sin x + x^2 - \dfrac{x^4}{6}$

$g'(x) = f'(x) + 2x - \dfrac{2x^3}{3}$

$g''(x) = f''(x) + 2x - 2x^2$

$g'''(x) = 3\sin x + x\cos x - 4x = 3(\sin x - x) + x(\cos x - 1)$

$\Rightarrow \quad g'''(x) < 0 \quad \Rightarrow \quad g''(x)$ is decreasing

for $x > 0 \quad g''(x) < g''(0) \Rightarrow \quad g''(x) < 0$

hence $g'(x)$ is decreasing

for $x > 0 \quad g'(x) < g'(0) \quad \Rightarrow \quad g'(x) < 0$

hence $g(x) < 0$

for $x > 0 \quad g(x) < g(0) \quad \Rightarrow \quad g(x) < 0$

Hence $f(x) < \dfrac{x^4}{6} - x^2 \ \forall\ x \in (0, \pi)$

SECTION – 2 : (Maximum Marks : 24)

- This section contains **EIGHT (8)** questions. The answer to each question is **NUMERICAL VALUE**.
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 6.25, 7.00, –0.33, –0.30,,30.27, –127.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : **+3** If ONLY the correct numerical value is entered as answer.

Zero Marks : **0** In all other cases.

7. The value of the integral $\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}} dx$ is _____ .

Ans. **(2)**

Sol. $\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$

$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$

put $\frac{1-x}{1+x} = t \quad \Rightarrow \quad \frac{-2dx}{(1+x)^2} = dt$

$I = \int_1^{\frac{1}{3}} \frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{t}}\right|_1^{1/3} = (1+\sqrt{3})(\sqrt{3}-1) = 2$

8. Let P be a matrix of order 3 × 3 such that all the entries in P are from the set {–1, 0, 1}. Then, the maximum possible value of the determinant of P is _____ .

Ans. **(4)**

Sol. det (P) = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 (b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \le 6$

value can be 6 only if $a_1 = 1$, $a_2 = -1$, $a_3 = 1$, $b_2c_3 = b_1c_3 = b_1c_2 = 1$, $b_3c_2 = b_3c_1 = b_2c_1 = -1$

$\Rightarrow \quad (b_2c_3)(b_3c_1)(b_1c_2) = -1 \quad \& \quad (b_1c_3)(b_3c_2)(b_2c_1) = 1$

i.e. $\quad b_1b_2b_3c_1c_2c_3 = 1$ and -1

hence not possible

Similar contradiction occurs when $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $b_2c_2 = b_3c_1 = b_1c_2 = 1$ $b_3c_2 = b_1c_3 = b_1c_2 = -1$

Now for value to be 5 one the terms must be zero but that will make 2 terms zero which means answer cannot be 5

Now $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 4$ Hence max value = 4

9. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto function form Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is ______ .

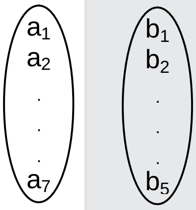
Ans. **(119)**

Sol. n(X) = 5

n(Y) = 7

$\alpha \rightarrow$ Number of one-one function = ${}^7C_5 \times 5! = 21 \times 120 = 2520$

$\beta \rightarrow$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

$$\frac{7!}{3!\,4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! = ({}^7C_3 + 3\,.\,{}^7C_3)\,5! = 4 \times {}^7C_3 \times 5!$$

$$\frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$$

10. Let f : R $\rightarrow$ R be a differentiable function with f(0) = 0. If y = f(x) satisfies the differential equation

$$\frac{dy}{dx} = (2 + 5y)(5y - 2),$$

then the value of $\lim_{x \to -\infty} f(x)$ is _______ .

Ans. **(0.4)**

Sol. $\frac{dy}{dx} = (5y + 2)(5y - 2)$

$$\frac{1}{25}\int \frac{dy}{\left(y + \frac{2}{5}\right)\left(y - \frac{2}{5}\right)} = \int dx$$

$$\frac{1}{25}.\frac{5}{4}\ell n\left|\frac{y - \frac{2}{5}}{y + \frac{2}{5}}\right| = x + c$$

$$\frac{1}{20}\ell n\left|\frac{5y - 2}{5y + 2}\right| = x + c$$

at x = 0, y = 0 $\Rightarrow$ c = 0

Hence $\frac{2 - 5y}{2 + 5y} = e^{20x}$

$$\frac{2 - 5y}{2 + 5y} = e^{20x}, \ \lim_{x \to \infty} e^{20x} = 0 \qquad \Rightarrow \qquad \lim_{x \to \infty} y = \frac{2}{5} = 0.4$$

11. Let $f : R \to R$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)\, f'(y) + f'(x) f(y) \text{ for all } x, y \in R.$$

Then, the value of $\log_e(f(4))$ is ______ .

Ans. **(2)**

Sol. $f(x + y) = f(x) . f'(y) + f'(x) . f(y)$

substituting $x = y = 0$, we get

$f(0) = 2f'(0) \quad \Rightarrow \quad f'(0) = \frac{1}{2}$

Now substituting $y = 0$

$f(x) = f(x) . f'(0) + f'(x) . f(0)$

$\Rightarrow \quad f'(x) = \frac{f(x)}{2}$

$\Rightarrow \quad f(x) = \lambda e^{x/2} \quad \Rightarrow \quad f(x) = e^{x/2}$ (as $f(0) = 1$)

Now $\ln(f(x)) = \frac{x}{2} \quad \Rightarrow \quad \ln (f(4)) = 2$

12. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _______ .

Ans. **(8)**

Sol. $P(\alpha, \beta, \gamma)$

$R(\alpha, \beta, -\gamma)$

Q

$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{0} = \frac{-2(\alpha+\beta-3)}{2}$

$x = 3 - \beta,\ y = 3 - \alpha,\ z = \gamma$

$Q(3 - \beta, 3 - \alpha, \gamma)$ lies on z –axis

$\therefore \quad \beta = 3, \alpha = 3$

$P(3, 3, \gamma)$ distance from x-axis is 5

$9 + \gamma^2 = 25$

$\gamma^2 = 16 \quad \Rightarrow \quad \gamma = 4$

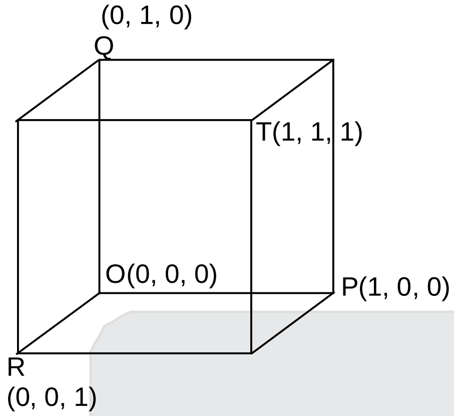
$P(3, 3, 4) \quad \therefore \quad PR = 8$

$R(3, 3, -4)$

13. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is ______ .

Ans. **(0.5)**

Sol.



point $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

point T(1, 1, 1)

$$\vec{p} = \overrightarrow{SP} = \frac{\hat{i}-\hat{j}-\hat{k}}{2}$$

$$\vec{q} = \overrightarrow{SQ} = \frac{-\hat{i}+\hat{j}-\hat{k}}{2}$$

$$\vec{r} = \overrightarrow{SR} = \frac{-\hat{i}-\hat{j}+\hat{k}}{2}$$

$$\vec{t} = \overrightarrow{ST} = \frac{\hat{i}+\hat{j}+\hat{k}}{2}$$

Now $\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix} \times \frac{1}{4} = \frac{1}{4}(2\hat{i}+2\hat{j}) = \frac{\hat{i}+\hat{j}}{2}$

$\vec{r}\times\vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \times \frac{1}{4} = \frac{-2\hat{i}+2\hat{j}}{4} = \frac{-\hat{i}+\hat{j}}{2}$

Now $(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \times \frac{1}{4} = \frac{\hat{k}}{2} \Rightarrow \left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right| = \frac{1}{2} = 0.5$

14. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots\ldots + 10({}^{10}C_{10})^2$ where ${}^{10}C_r$, $r \in \{1, 2, \ldots\ldots, 10\}$ denote binomial coefficients. Then the value of $\frac{1}{1430}X$ is _______ .

Ans. **(646)**

Sol. $X = \sum_{r=1}^{10} r.\,{}^{10}C_r.{}^{10}C_r = 10.\sum_{r=1}^{10} {}^{9}C_{r-1}.{}^{10}C_{10-r} = 10\,.\,{}^{19}C_9$

Now $\frac{X}{1430} = \frac{10.{}^{19}C_9}{1430} = \frac{{}^{19}C_9}{143} = \frac{{}^{19}C_9}{11\times 13} = \frac{19.17.16}{8} = 19 \times 34 = 646$

SECTION 3 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **TWO (02)** matching lists: **LIST-I** and **LIST-II**.
- **FOUR** options are given representing matching of elements from **LIST-I** and **LIST-II**. **ONLY ONE** of these four options corresponds to a correct matching.
- For each question, choose the option corresponding to the correct matching.
- For each question, marks will be awarded according to the following marking scheme:

  Full Marks : 3 If ONLY the option corresponding to the correct matching is chosen.

  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).

  Negative Marks : 1 In all other cases.

15. Let $E_1 = \left\{x \in R : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

Let $f : E_1 \to R$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to R$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$

LIST-I	LIST-II
(P) The range of f is	(1) $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
(Q) The range of g contains	(2) (0, 1)
(R) The domain of f contains	(3) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(S) The domain of g is	(4) $(-\infty, 0) \cup (0, \infty)$
	(5) $\left(-\infty, \frac{e}{e-1}\right]$
	(6) $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is

(A) P → 4; Q → 2; R → 1; S → 1

(B) P → 3; Q → 3; R → 6; S → 5

(C) P → 4; Q → 2; R → 1; S → 6

(D) P → 4; Q → 3; R → 6; S → 5

Ans. **(A)**

Sol. $E_1 : \frac{x}{x-1} > 0 \quad \Rightarrow \quad x \in (-\infty, 0) \cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\frac{x}{x-1}\right) \le 1 \quad \Rightarrow \quad \frac{1}{e} \le \frac{x}{x-1} \le e \quad \Rightarrow \quad \frac{1}{e} \le 1 + \frac{1}{x-1} \le e$

$\frac{1}{e}-1 \le \frac{1}{x-1} \le e-1 \quad \Rightarrow \quad (x-1) \in \left(-\infty, \frac{e}{1-e}\right] \cup \left[\frac{1}{e-1}, \infty\right)$

$x \in \left(-\infty, \frac{1}{e-1}\right] \cup \left[\frac{e}{e-1}, \infty\right)$

Now $\frac{x}{x-1} \in (0, \infty) - \{1\} \ \forall\ x \in E_1 \quad \Rightarrow \quad \ell n\left(\frac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$

$\sin^{-1}\left(\ell n\left(\frac{x}{x-1}\right)\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

16. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.

(ii) Let α_2 be the total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST-I	**LIST-II**
(P) The value of α_1 is	(1) 136
(Q) The value of α_2 is	(2) 189
(R) The value of α_3 is	(3) 192
(S) The value of α_4 is	(4) 200
	(5) 381
	(6) 461

The correct option is

(A) $P \to 4$; $Q \to 6$; $R \to 2$; $S \to 1$

(B) $P \to 1$; $Q \to 4$; $R \to 2$; $S \to 3$

(C) $P \to 4$; $Q \to 6$; $R \to 5$; $S \to 2$

(D) $P \to 4$; $Q \to 2$; $R \to 3$; $S \to 1$

Ans. **(C)**

Sol. 6 Boys & 5 girls

$\alpha_1 \to$ number of ways of selecting 3 boys & 2 girls $^6C_3 \times {}^5C_2 = 200$

$\alpha_2 \rightarrow$ Boys & girls are equal & members ≥ 2

${}^6C_1 . {}^5C_1 + {}^6C_2 . {}^5C_2 + {}^6C_3 . {}^5C_3 + {}^6C_4 . {}^5C_4 + {}^6C_5 . {}^5C_5 = {}^{11}C_5 - 1 = 461$

$\alpha_3 \rightarrow$ number of ways of selecting 5 having at least 2 girls ${}^{11}C_5 - {}^6C_5 - {}^6C_4 . {}^5C_1 = {}^{11}C_5 - 81 = 381$

$\alpha_4 \rightarrow G_1$ is included $\rightarrow {}^4C_1 . {}^5C_2 + {}^4C_2 . {}^5C_1 + {}^4C_3 = 40 + 30 + 4 = 74$

M_1 is included $\rightarrow {}^4C_2 . {}^5C_1 + {}^4C_3 = 34$

G_1 & M_1 both are excluded $\rightarrow {}^4C_4 + {}^4C_3 . {}^5C_1 + {}^4C_2 . {}^5C_2 = 81$

Total = 74 + 34 + 81 = 189

17. Let H : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I	**LIST-II**
(P) The length of the conjugate axis of H is	(1) 8
(Q) The eccentricity of H is	(2) $\frac{4}{\sqrt{3}}$
(R) The distance between the foci of H is	(3) $\frac{2}{\sqrt{3}}$
(S) The length of the latus rectum of H is	(4) 4

The correct option is:

(A) P $\rightarrow$ 4; Q $\rightarrow$ 2; R $\rightarrow$ 1; S $\rightarrow$ 3

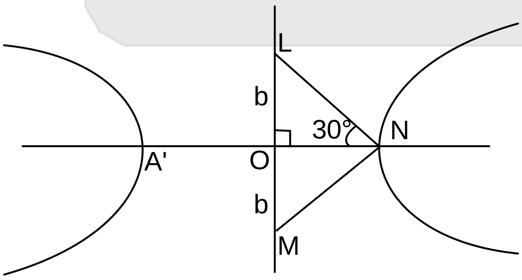
(B) P $\rightarrow$ 4; Q $\rightarrow$ 3; R $\rightarrow$ 1; S $\rightarrow$ 2

(C) P $\rightarrow$ 4; Q $\rightarrow$ 1; R $\rightarrow$ 3; S $\rightarrow$ 2

(D) P $\rightarrow$ 3; Q $\rightarrow$ 4; R $\rightarrow$ 2; S $\rightarrow$ 1

Ans. **(B)**

Sol.



Area of LMN = $4\sqrt{3}$

$\frac{1}{2}(2b)(\sqrt{3}b) = 4\sqrt{3} \Rightarrow b^2 = 4 \Rightarrow b = 2 \Rightarrow 2b = 4$

Here $\frac{a}{b} = \cot 30°$ $\Rightarrow$ $a = \sqrt{3}b \Rightarrow$ $a = 2\sqrt{3}$

$b^2 = a^2 (e^2 - 1)$

$4 = 12(e^2 - 1)$

$e^2 = 1 + \frac{1}{3} = \frac{4}{3} \Rightarrow e = \frac{2}{\sqrt{3}}$ and $2ae = 2 \times 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 8$

and length of latus ractum = $\frac{2b^2}{a} = \frac{2 \times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

18. Let $f_1 : R \to R$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to R$ and $f_4 : R \to R$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $f_3(x) = [\sin(\log_e(x+2))]$, where for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

LIST-I	**LIST-II**
(P) The function f_1 is	(1) **NOT** continuous at x =0
(Q) The function f_2 is	(2) continuous at x = 0 and **NOT** differentiable at x = 0
(R) The function f_3 is	(3) differentiable at x = 0 and its derivative is **NOT** continuous at x = 0
(S) The function f_4 is	(4) differentiable at x= 0 and its derivative is continuous at x= 0

The correct option is:

(A) P → 2; Q → 3; R → 1; S → 4

(B) P → 4; Q → 1; R → 2; S → 3

(C) P → 4; Q → 2; R → 1; S → 3

(D) P → 2; Q → 1; R → 4; S → 3

Ans. **(D)**

Sol. (i) $f'_1(0) = \lim\limits_{h\to 0} \dfrac{\sin\sqrt{1-e^{-h^2}} - 0}{h} = \lim\limits_{h\to 0} \dfrac{\sin\sqrt{1-e^{-h^2}}}{\sqrt{1-e^{-h^2}}} \times \sqrt{\dfrac{1-e^{-h^2}}{h^2}} \times \dfrac{|h|}{h}$

$= 1 \times 1 \times \dfrac{|h|}{h} = 1 \times 1 \times \dfrac{|h|}{h}$

= limit does not exist.

⇒ for option (P), (2) is correct.

(ii) $\lim\limits_{x\to 0} f_2(x) = \lim\limits_{x\to 0} \dfrac{|\sin x|}{\tan^{-1} x}$

$= \lim\limits_{x\to 0} \dfrac{|\sin x|}{|x|} \times \dfrac{x}{\tan^{-1} x} \times \dfrac{|x|}{x}$

$= \lim\limits_{x\to 0} 1 \times 1 \times \dfrac{|x|}{x}$

= limit does not exist ⇒ for option Q, (1) is correct.

(iii) $\lim\limits_{x\to 0} f_3(x) = \lim\limits_{x\to 0} [\sin(\log_e(x+2))]$

now at x tends to zero (x + 2) tends to 2

⇒ $\log_e(x + 2)$ tends to $\ell n2$

which is less than 1

$0 < \lim\limits_{x\to 0} \sin(\log_e(x + 2)) < \sin 1$ ⇒ $\lim\limits_{x\to 0} [\sin(\log_e(x + 2))] = 0$

$f_3(x) = \{0 \quad x \in [-1, e^{\pi/2} - 2)$

⇒ $f'_3(x) = 0 \quad \forall x \in (-1, e^{\pi/2} - 2)$

⇒ $f''_3(x) = 0 \quad \forall x \in (-1, e^{\pi/2} - 2)$

Hence for (R), (4) is correct.

(iv) $\lim\limits_{x\to 0} f_4(x) = \lim\limits_{x\to 0} \left(x^2 \sin\dfrac{1}{x}\right) = \lim\limits_{x\to 0} x^2\left(\sin\dfrac{1}{x}\right) = 0$

$f'_4(0) = \lim\limits_{h\to 0} \dfrac{h^2 \sin\left(\dfrac{1}{x}\right) - 0}{x} = \lim\limits_{h\to 0} h\sin\left(\dfrac{1}{x}\right) = 0$

$f'_4(x) = -\cos\dfrac{1}{x} + x\sin\dfrac{1}{x}, \ x \neq 0$

$f''_4(0) = \dfrac{-\cos\dfrac{1}{h} + h\sin\dfrac{1}{h} - 0}{h}$ ⇒ does not exist

hence for (S), (3) is correct.