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INDIAN OLYMPIAD QUALIFIER (IOQ) 2022-2023

INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS (IOQM), 2022

QUESTION PAPER WITH SOLUTION

Resonance Eduventures Ltd.

Reg. Office & Corp. Office : CG Tower, A-46 & 52, IPIA, Near City Mall, Jhalawar Road, Kota (Raj.) - 324005

Ph. No.: +91-744-2777777, 2777700 | **FAX No. :** +91-022-39167222

To Know more : sms **RESO** at **56677** | **Website :** www.resonance.ac.in | **E-mail :** contact@resonance.ac.in | **CIN :** U80302RJ2007PLC024029

Toll Free : 1800 258 5555 7340010333 facebook.com/ResonanceEdu twitter.com/ResonanceEdu www.youtube.com/resowatch blog.resonance.ac.in

Time : 3 hrs **October 30, 2022** **Total marks : 100**

INSTRUCTIONS

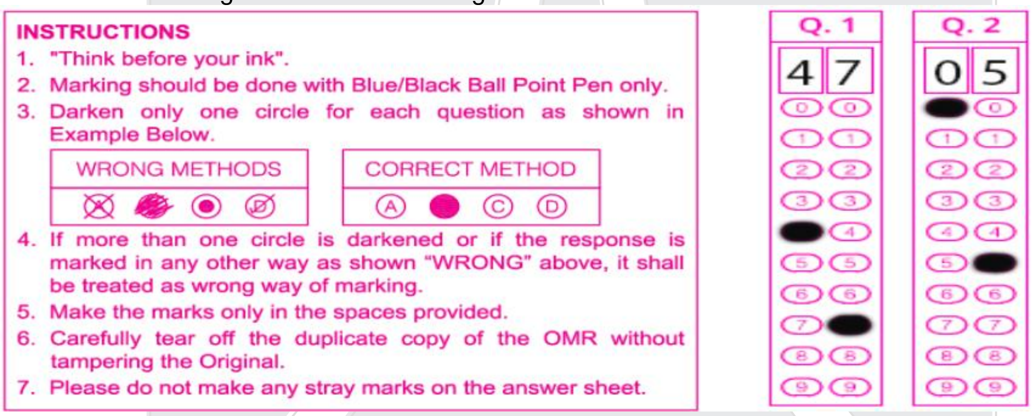
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer. .

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 10 carry 2 marks each; questions 11 to 22 carry 5 marks each; questions 23 and 24 carry 10 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

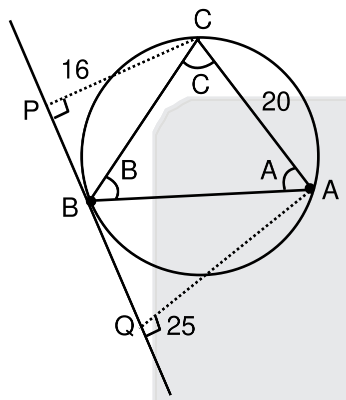
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. A triangle ABC with AC = 20 is inscribed in a circle w. A tangent t to w is drawn through B. The distance of t form A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding S/20.

Ans. 10

Sol.



$\angle ABQ = C$

$\angle PBC = A$

now $\sin C = \dfrac{AQ}{AB} = \dfrac{25}{AB}$

& $\sin A = \dfrac{PC}{BC} = \dfrac{16}{BC}$

Area of ΔABC

$= \dfrac{1}{2}.AB.AC.\sin A$

$= = \dfrac{1}{2}.20.AB\sin A$

$= 10.\dfrac{25}{\sin c}.\sin A$

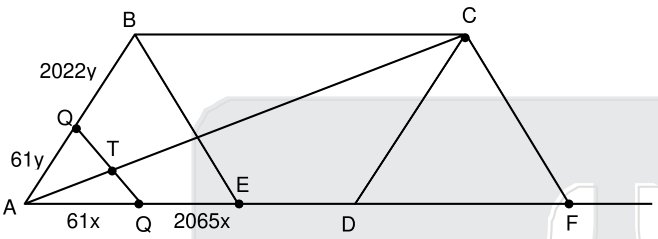
= 200

So $\dfrac{S}{20} = 10$

2. In a parallelogram ABCD, a point P on the segment AB is taken such that $\frac{AP}{AB} = \frac{60}{2022}$ and a point Q on the segment AD is taken such that $\frac{AQ}{AD} = \frac{60}{2065}$. If PQ intersects AC at T, find $\frac{AC}{AT}$ to the nearest integer.

Ans. **67**

Sol.



Draw BE|| to PQ

Draw CF|| to PQ

$\Delta AQT \sim \Delta AFC$

and BCFE is a parallelogram

EF = BC = AD

Now $\frac{AP}{PB} = \frac{AQ}{QE}$

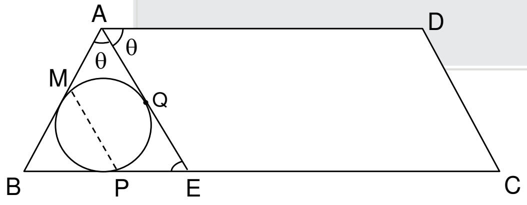
So, QE = 1961x

now, $\frac{AC}{AT} = \frac{AF}{AQ} = 67$

3. In a trapezoid ABCD, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if AB : MP = 2.

Ans. **60**

Sol.



Now ∠ BAE = ∠DAE = θ (say)

also ∠ BAE = ∠BEA = θ (lines parallel)

So AM = AQ = EQ = EP

So MP || AE

also AB = AE

also M is mid point of AB

& P is mid point of BE

So $\frac{AB}{MP} = \frac{AE}{MP} = 2$

So Δ ABE is equilateral Δ.

So θ = 60º

4. Starting with a positive integer M written on the board, Alice plays the following game: in each more, if x is the number on the board, she replaces it with 3x + 2. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with 2x + 27. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of M + N.

Ans. **10**

Sol. For Alice , $x \to 3x + 2$

In 4 Trials, if will be as

$(3m + 2)$, $3(3M + 2) + 2$,

$9(3M + 2) + 8$, $27(3M + 2) + 26$

4th Trial - He will be at

$81M + 80$

for Bob $N \to 2N + 27$

In 4 trials, it will be as

$2N + 27$, $2(2N + 27) + 27$,

$4(2N + 27) + 81$, $8(2N+27) + 189$.

According to questions, they both get same number,

So, $81M + 80 = 16N + 15 \times 27$

$81M - 16N = 405 - 80 = 325$

$M_{(min)} = 5$

$N_{(min)} = 5$

$M + N = 10$

5. Let m be the smallest positive integer such that $m^2 + (m + 1)^2 + \ldots\ldots + (m + 10)^2$ is the square of a positive integer n. Find m + n.

Ans. **95**

Sol. $m^2 + (m+1)^2 + (m+2)^2 + \ldots\ldots + (m+10)^2 = n^2$

$11m^2 + m(2.1 + 2.2 + \ldots.. + 2.10) + (1^2 + 2^2 + \ldots\ldots + 10^2) = n^2$

$11m^2 + 110m + 35 \times 11 = n^2$

So $n = 11k$ (multiple of 11)

$11.(m^2 + 110m + 35) = 11K.11k$

$m^2 + 10m + 35 = 11K^2$

Now $m^2 + 10m + 35 - 11K^2 = 0$

$$m = \frac{-10 \pm \sqrt{4(11K^2 - 10)}}{2}$$

$= -5 \pm \sqrt{11K^2 - 10}$

$D = 100 - 4x1x(35 - 11K^2)$

$= 100 - 140 + 44K^2$

$= 44K^2 - 40$

$= 4(11K^2 - 10)$ is a perfect square

putting $K = 7$

So $m+n = 18+77 = 48$

6. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

Ans. **43**

Sol. $a^3 - b^3 - ab = 25$

largest value of $a^2 + b^3$

clearly a > b

a & b both natural numbers

let b = 1: $a^3 - a = 26 \rightarrow$ no solution

b = 2: $a^3 - 2a = 33 \rightarrow$ no solution

b = 3: $a^3 - 3a = 52 \rightarrow a = 4$

b = 4: $a^3 - 4a = 89 \rightarrow$ no. solution

considering $1 \le a^2 + b^3 \le 99$

value of b only 1, 2, 3, 4 only

so we have a = 4 and b = 3

so $a^2 + b^3 = 4^2 + 3^3$

$= 16 + 27 = 43$

7. Find the number of ordered pairs (a, b) such that a, b $\in$ {10, 11., 29, 30} and GCD (a, b) + LCM (a, b) = a + b.

Ans. **35**

Sol. GCD (a, b) + LCM (a, b) = a+b

Case - I $a = b \in \{10, 11, 12, \ldots\ldots 30\}$

$\Rightarrow$ 21 pairs

Case - II One of a or b is the multiple of other and $a \neq b$

$\Rightarrow$ (10, 20), (10, 30), (11, 22), (12, 24) (13, 26), (14, 28), (15, 30)

$\Rightarrow$ ordered pairs = 7 × 2 = 14

Total no. of ordered pairs = 21 + 14 = 35

8. Suppose the prime number p and q satisfy $q^2 + 3p = 197p^2 + q$. Write $\frac{q}{p}$ as $l + \frac{m}{n}$, where l, m, n are positive integers, m < n and GCD (m,n) = 1. Find the maximum value of l + m + n.

Ans. **32**

Sol. $q^2 + 3p = 197p^2 + q$

$q^2 - q = 197p^2 - 3p$

$q(q - 1) = p(197p - 3)$

Let $q - 1 = kp \Rightarrow q = kp + 1$

$q.kp = p(197P - 3)$

$qk = 197p - 3$

$(kp + 1)k = 197P - 3$

$k^2P + K = 197P - 3$

$k^2p - 197P + K + 3 = 0$

$(k^2 - 197)P = -(k + 3)$

$$p = \frac{K+3}{197-K^2}$$

K = 14

So, P = 17

$q = 14 \times 17 + 1$

$= 239$

$$\frac{q}{p} = \frac{239}{17} = 14 + \frac{1}{17}$$

$\ell = 14$

m = 1

n = 17

$\ell + m + n = 32$

9. Two sides of an integer sided triangle have lengths 18 and x. If there are exactly 35 possible integer values y such that 18, x, y are the sides of a non- degenerate triangle, find the number of possible integer values x can have.

Ans. **(Bonus)**

Sol. For a non-degenerate triangle

$x + y > 18$ and $18 + x > y$

$y > 18 - x$ $\quad$ $y < 18 + x$

$18 - x < y < 18 + x$ …….(1)

$x = 1$; $17 < y < 19$; $y = 18$

$x = 2$; $16 < y < 20$; $y = 17, 19$

$x = 3$ $\quad$ $15 < y < 21$ $\quad$ $y = 16, 20$

⋮ ⋮ ⋮

$x = 18$ $\quad$ $0 < y < 36$ $\quad$ $y = 1, 35$

but x can be any integer which is greater than equal to 18

10. Consider the 10-digit number M= 9876543210. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from M = $9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get M_1 = $9\overline{78}6\overline{45}3210$ .Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

Ans. **88**

Sol. Let there are n digits in m

when one pair of two digits is selected $= n - 1 = 9$

when two pair of two digits is selected $= (n - 3) + (n - 4) + \ldots\ldots 1$

$$= \frac{(n-3)(n-2)}{2} = 28$$

3 pairs of two digits is selected

$$= \frac{(8-3)(8-2)}{2} + \frac{(7-3)(7-2)}{2} + \frac{(6-3)(6-2)}{2} + \frac{(5-3)(5-2)}{2} + \frac{(4-3)(4-2)}{2}$$

$= 15 + 10 + 6 + 3 + 1 = 35$

4 pairs of two digits is selected

$$= \frac{(6-3)(6-2)}{2} + \frac{(5-3)(5-2)}{2} + \frac{(4-3)(4-2)}{2}$$

$$+ \frac{(5-3)(5-2)}{2} + \frac{(4-3)(4-2)}{2} + 1 = 6 + 3 + 1 + 3 + 2 + 1 = 16$$

5 points of two digits is selected $= 1$

Answer : $9 + 28 + 35 + 16 + 1 - 1 = 88$

11. Let AB be a diameter of a circle w and let C be a point on w, different from A and B. The perpendicular from C intersects AB at D and w at E(≠C). The circle with centre at C and radius GD intersects w at P and Q . if the perimeter of the triangle PEQ is 24, find the length of the side PQ.

Ans. 08

Sol. Take C as mid point of arc AB

CP = CD = CQ = AD = DB as all radius

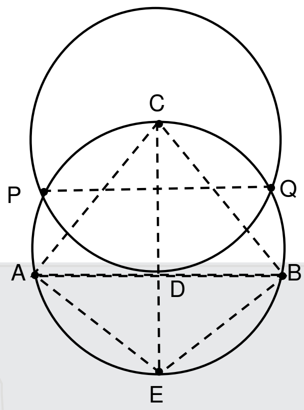
PQ is common chord.

E is diametrically opposite point of C

Now PE + EQ + PQ = 24 given

By Symmetry Δ PEQ is equilateral

So PQ = 8



12. Given ΔABC with ∠B = 60º and ∠G = 30º, let P, Q, R be points on sides BA, AC, CB respectively such that BPQR is an isosceles trapezium with PQ ||BR and BP = QR. Find the maximum possible value of $\frac{2[ABC]}{[BPQR]}$ where [S] denotes the area of any polygon S.

Ans. **(Bonus)**

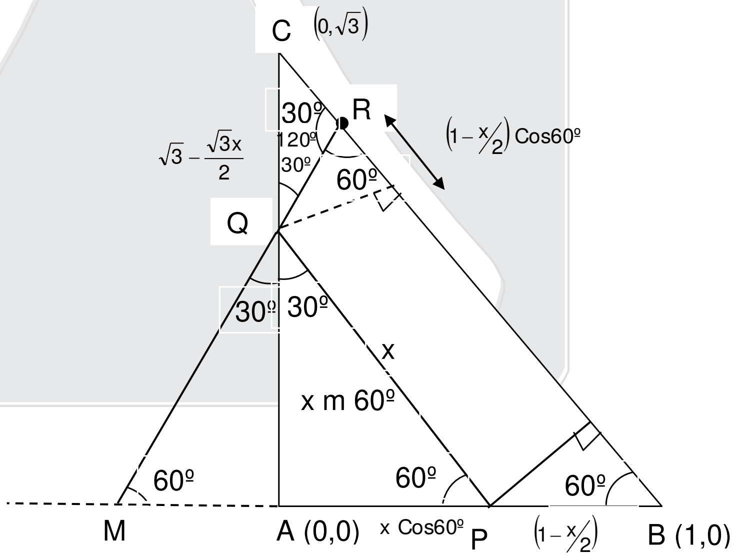
Sol. Area of Trapezium

$$=\frac{1}{2}\left(x+x+\left(1-\frac{x}{2}\right)\text{Cos}60^\circ\times 2\right)$$

$$\left(1-\frac{x}{2}\right)\text{Cos}60^\circ$$

$$=\frac{1}{2}\left(2x+\left(1-\frac{x}{2}\right)\right)\left(1-\frac{x}{2}\right).\frac{\sqrt{3}}{2}$$

$$=\frac{1}{2}\left(\frac{3x}{2}+1\right)\left(1-\frac{x}{2}\right).\frac{\sqrt{3}}{2}$$



Area = $\frac{\sqrt{3}}{16}(3x+2)(2-x)$

$= \frac{\sqrt{3}}{16}(-3x^2+4\times+4)$ (maximum value at x = $\frac{2}{3}$)

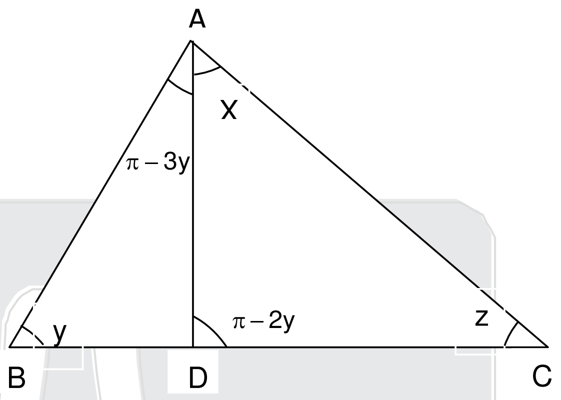
$\frac{2[ABC]}{[BPQR]} = \frac{2\times\frac{1}{2}\times 1.\sqrt{3}}{\frac{1}{\sqrt{3}}}$ = 3 (this is minimum value maximum is not possible)

13. Let ABC be a triangle and let D be a point on the segment BC such that AD = BC. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

Ans. 59

Sol. Let $x = y - \alpha$

$z = y + \alpha$



In Δ ADC

$$\frac{CD}{\sin(y-\alpha)} = \frac{AD}{\sin(y+\alpha)}$$

$$CD = AD\,\frac{\sin(y-\alpha)}{\sin(y+\alpha)} \quad \text{\_\_\_\_\_\_(1)}$$

In Δ ABD

$$\frac{BD}{\sin(\pi-3y)} = \frac{AD}{\sin y}$$

$$BD = AD\,\frac{\sin 3y}{\sin y} \quad \text{\_\_\_\_\_\_(2)}$$

(1) + (2)

$$CD + BD = AD\left(\frac{\sin 3y}{\sin y} + \frac{\sin(y-\alpha)}{\sin(y+\alpha)}\right)$$

$$1 = 3 - 4\sin^2 y + \frac{\sin(y-\alpha)}{\sin(y+\alpha)}$$

$$\frac{4\sin^2 y - 2}{1} = \frac{\sin y\cos\alpha - \cos y\sin\alpha}{\sin y\cos\alpha + \cos y\sin\alpha}$$

$$\frac{4\sin^2 y - 1}{4\sin^2 y - 3} = \frac{2\sin y\cos\alpha}{2\cos y\sin\alpha} \Rightarrow \frac{3-4\cos^2 y}{4\sin^2 y - 3} = \frac{\sin y\cos\alpha}{-\cos y\sin\alpha}$$

$$\frac{\cos 3y}{\sin 3y} = -\frac{\cos\alpha}{\sin\alpha}$$

$\tan 3y = -\tan\alpha$

$\tan 3y = \tan(\pi - \alpha)$

$3y = \pi - \alpha$

$\alpha = \pi - 3y$

$x = y - \alpha = 4y - \pi > 0 \Rightarrow y > \pi/4$

$z = y + \alpha = \pi - 2y > 0 \Rightarrow y < \pi/2$

$\alpha = \pi - 3y > 0 \Rightarrow y < 60$

$y \in (45, 60)$

maximum vq/ve of y = 59

14. Let x,y, z be complex numbers such that

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=9$$

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=64$$

$$\frac{x^3}{y+z}+\frac{y^3}{z+x}+\frac{z^3}{x+y}=488$$

If $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}=\frac{m}{n}$ where m,n are positive integers with GCD (m,n) = 1, Find m + n.

Ans. 16

Sol. Given equation (1) gives on solving

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=9$$

add 3

$$1+\frac{x}{y+z}+1+\frac{y}{z+x}+1+\frac{z}{x+y}=12$$

$$(x+y+z)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)=12$$

i.e. $(x+y+z)\sum\frac{1}{x+y}=12$

Now multiply (i) by (x+y+z) we get

$$(x+y+z)\sum\frac{x}{y+z}=9(x+y+z)$$

i.e. $\sum\frac{x^2}{y+z}+(x+y+z)=9(x+y+z)$

i.e 64 + (x+y+z) = 9 (x+y+z)

(x+y+z) = 8

now (x+y+z) $\sum\frac{x^2}{y+z}=64(x+y+z)$

i.e. $\sum\frac{x^3}{y+z}+x^2+y^2+z^2=64\times 8$

ie. 488 + x^2 + y^2 + z^2 = 64 × 8

i.e x^2 + y^2 + z^2 = 24

so we have x + y + z = 8

& $x^2+y^2+z^2=24$

& $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{3}{2}$

from above we get xyz = 104

so $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}=\frac{3}{13}$

so m+n = 16

15. Let x, y be real numbers such that xy =1 . Let T and t be the largest and the smallest values of the expression.

$$\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}$$

If T + t can be expressed in the form $\frac{m}{n}$ where m,n are nonzero integers with GCD (m,n) = 1 , find the value of m + n

Ans. **25**

Sol. x.y = 1

$$\exp = \frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}$$

$$= = \frac{(x-y)^2+4xy-(x-y)-2}{(x-y)^2+4xy+(x-y)-2},\ x-y=p\in R$$

Let, $z = \frac{p^2-p+2}{p^2+p+2}$

$p^2(z-1)+p(z+1) + 2(z-1) = 0$

$7z^2 - 18z + 7 \leq 0$

$\frac{9-4\sqrt{2}}{7}$ $\frac{9-4\sqrt{2}}{7}$

$y_{min} = t = \frac{9-4\sqrt{2}}{7}$

$y_{max} = T = \frac{9+4\sqrt{2}}{7}$

Now, $T + t = \frac{18}{7} = \frac{m}{n}$

$\therefore m + n = 25$

16. Let a, b, c be reals satisfying

3ab + 2 = 6b , 3bc + 2 = 5c , 3ca + 2 = 4a.

Let Q denote the set of all rational numbers. Given that the product abc can take two values $\frac{r}{s}\in Q\ \&\ \frac{t}{u}\in Q$, in lowest form, find r + s + t + u.

Ans. **18**

Sol. 3ab + 2 = 6b,….(i)

3bc + 2 = 5c……(ii)

3ac+2=4a…….(iii)

Multiply (i) with c $\Rightarrow$ 3abc + 2c = 6bc

$\Rightarrow$ 3abc + 2c = 2(5c–2)

$\Rightarrow$ 3abc + 2c = 10c – 4

$\Rightarrow$ 9abc = 24c– 12

Multiply (ii) with a $\Rightarrow$ 3abc + 2a = 5ac

$\Rightarrow 3abc + 2a = 5\left(\frac{4a-2}{3}\right)$

$\Rightarrow$ 9abc + 6a = 20a–10

$\Rightarrow$ 9abc = 14a–10

Multiply (iii) with b $\Rightarrow$ 3abc + 2b = 4ac

$\Rightarrow 3abc + 2b = 4\left(\frac{6b-2}{3}\right)$

$\Rightarrow 9abc + 6b = 24b–8$

$\Rightarrow 9abc = 18b– 8$

hence $24c – 12 = 14a–10 = 18b–8 = k$ (let)

$a = \frac{10+k}{14}, b = \frac{k+8}{18}, c = \frac{k+12}{24}$

Now from (i) $3ab + 2 = 6b$

$3\left(\frac{k+10}{14}\right)\left(\frac{k+8}{18}\right) + 2 = 6\left(\frac{k+8}{18}\right)$

$k^2 – 10k + 24 = 0 \Rightarrow k = 6, 4$

when $k = 6 \Rightarrow a = \frac{8}{7} \Rightarrow abc = \frac{2}{3}$

when $k = 4 \Rightarrow a = 1 \Rightarrow abc = \frac{4}{9}$

$r + s + t + u = 18$

17. For a positive integer n > 1, let g(n) denotes the largest positive proper divisor of n and f(n) = n – g(n).For example , g(10)= 5, f(10) = 5 g(13) = 1, f(13) = 12. Let N be the smallest positive integer such that f(f(f(N))) = 97. Find the largest integer not exceeding $\sqrt{N}$

Ans. 24

Sol f (n) = n – g (n) g (n) = Largest positive proper divisor of n

f (2) = 2 – g (2) = 1, g (2) = 1

f (3) = 3 – g (3) = 2, g (3) = 1

f (4) = 4 – g (4) = 2, g (4) = 2

f (5) = 5 – g (5) = 4, g (5) = 1

f (6) = 6 – g (6) = 3, g (6) = 3

f (7) = 7 – g (7) = 6, g (7) = 1

f (8) = 8 – g (8) = 4, g (8) = 4

Hence $f(K) = \begin{cases} K/2 & K = \text{even} \\ K-1 & K = \text{odd prime} \end{cases}$

Now f (f (f (N))) = 97 Let f (f (N)) = K

f (K) = 97 ⇒ K = 194 ⇒ f (f (N)) = 194 Let f (N) = λ

⇒ f (λ) = 194

⇒ λ = 291

⇒ f (N) = 291

⇒ N = 582

⇒ $\sqrt{N}$ = 24.12

Largest integer not exceeding $\sqrt{N}$ is 24

18. Let m, n be natural numbers such that m + 3n – 5 = 2 LCM (m,n) – 11GCD (m,n).
Find the maximum possible value of m + n.

Ans. **70**

Sol. m + 3n – 5 = 2 LCM (m,n) – 11 GCD (m,n)

C-1 when m = even, n = even
L.C.M. (m, n) = even
G.C.D. (m,n) = even
L.H.S. odd , R.H.S. = even not possible

C-2
when m = odd , n = odd
L.H.S. = odd , R.H.S. = odd
(a) when m = 1, L.C.M. (1,n) = n , G.C.D. (1, n) = 1
m + 3n – 5 = 2n – 11
m + n = – 6 not possible m,n are natural
(b) when m,n both are prime
then L.C.M. (m,n) = mn
GCD (m,n) = 1
m + 3n – 5 = 2mn – 11 $\Rightarrow$ 3n + 6 = m (2n – 1)
m + 3n + 6 = 2mn

$$2m = 3 + \frac{15}{2n-1}$$

2n – 1 = 1, 3, 5, 15
2n = 2, 4, 6, 16
n = 1, 2, 3, 8
2m = 18, 8, 6, 4
m = 9, 4, 3, 2
m + n = 6
(c)
when n is multiple of m (m < n)
L.C.M (m,n) = n
G.C.D (m,n) = m
m + 3n – 5 = 2n – 11m
12m + n = 5
not possible
(d) when m is multiple of n (n < m)
LCM (n,m) = m
GCD (m,n) = n
m + 3n – 5 = 2m – 11n
m – 14n = – 5   m = 14n – 5
n = 5,   m = 65
C-3 when m = even , n = odd
L.H.S = even R.H.S. = odd
Ans. m = 65, n = 5
m + n = 70

19. Consider a string of n 1' s. we wish to place some + signs in between so that the sum is 1000. for instance, If n = 190, one may put + signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of a.

Ans. **10**

Sol. Number of ways of taking n will be as following

$11 \times 90 + 1 \times 10$ times
$11 \times 89 + 1 \times 21$ times
⋮
$11 \times 1 + 1 \times 989$ times
} 90 terms

$111 \times 9 + 1 \times 1$
⋮
$111 \times 1 \quad + \times 89$
} 9 terms

$111 \times 8 + 11 \times 10 + 1 \times 2$
$111 \times 8 + 11 \times 9 + 1 \times 13$
⋮
$111 \times 8 + 11 \times 1 + 1 \times 101$
} 10 terms

$111 \times 7 + 11 \times 20 + 1 \times 3$
⋮
$111 \times 7 + 11 \times 1 + 1 \times 212$
} 20 terms

similarly

$111 \times 1 + 11 \times 80 + 1 \times 9$
⋮
$111 \times 1 + 11 \times 1 + 1 \times 878$
} 80 terms

1×1000} 1 terms

total possible ways = a = 10 + 20 + 30 +........ + 90 + 9 +1

$$= = \frac{9}{2}[10+90]+9+1$$

$$= 9\times 50+9+1$$

$$= 450 + 9+1 = 460$$

sum of digits of a = 10

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots\ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say p_1 is a landmark point if $2 \leq \ell \leq n-1$ and $(p_\ell - 1 - p_\ell > 0$. for example, for $n = 7$, the permutation (2,7, 6, 4, 5, 1, 3) has four landmark points : $p_2 = 7$, $p_4 = 4$, $p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let L(n) denote the number of permutations of $\{1, 2, \ldots, n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which L(n) is a perfect square

Ans. **03**

Sol. $n \geq 3$ $\sigma(p_1, p_1 \ldots\ldots p_n)$ & $\{1,2,3,.., n\}$

p_ℓ where $2 \leq \ell \leq n-1$

$(p_{\ell-1} - p_\ell)(p_{\ell+1} - p_\ell) > 0$

both +ve $p_{\ell+1} > p_\ell$ & $p_{\ell-1} > p_\ell$

both –ve $p_{\ell-1} > p_\ell$ & $p_{\ell+1} > p_\ell$

i.e. p_ℓ will be a landmark point if

both $p_{\ell+1}$ & $p_{\ell-1}$ are larger than it of both less than it

L(n) = number of permutation of 1, 2, …, n

with only one landmark point to find maximum n for which L(x) is perfect square

if p_ℓ is not a landmark point then it is

either increasing $(p_{\ell-1} < p_\ell < p_{\ell+1})$ or

Decreasing $(p_{\ell-1} > p_\ell > p_{\ell+1})$

so the series is either increasing decreasing – case1

or decreasing increasing – case2

In case-1 $\rightarrow$ the maximum number n

is at 2nd place in ${}^{n-1}C_1$ ways & so on

so total ${}^{n-1}C_1 + {}^{n-1}C_2 \ldots\ldots\ldots + {}^{n-1}C_{n-2}$

In case -2 $\rightarrow$ the minimum number 1 is at 2nd place in ${}^{n-1}C_1$ ways & so on

so total ${}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots + {}^{n-1}C_{n-2}$

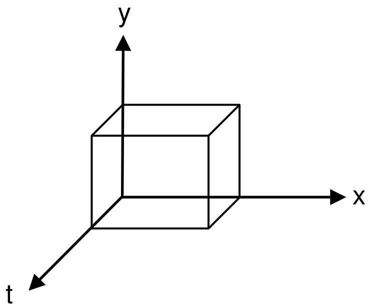
so $L(x) = 2\,({}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots + {}^{n-1}C_{n-2} = 2^{n-1} - 2)$

for n = 3 L(x) is perfect square

so n = 3

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge if N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N.

Ans. **74**



Sol. Along one axis $\rightarrow$ 3 ways = 3

along two axis $\rightarrow {}^3C_2 . \dfrac{6!}{4!2!} = 45$

along there axis $\rightarrow {}^3C_3 . \dfrac{6!}{2!2!2!} = 90$

$= 138$

so Ans. $= 1^2 + 3^2 + 8^2 = 74$

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called friendly if each term is adjacent to at least one term that is equal to 1. for example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

Ans. **10**

Sol. $P_1 = 1$

$P_2 = (01, 11, 10) = 3$

$$P_3 = \begin{pmatrix} 111 \\ 011 \\ 110 \\ 101 \\ 010 \end{pmatrix} = 5$$

$P_4 = 8$

we get $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

we have

$P_5 = 13$

$P_6 = 21$

$P_7 = 34$

$P_8 = 55$

$P_9 = 89$

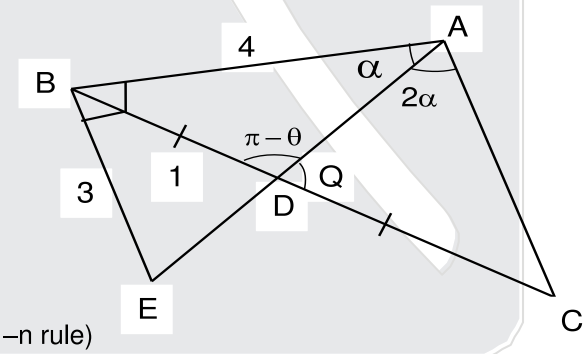
$P_{10} = 144$

so minimum n so $F_n > 100$ is n = 10

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ration 1:2. Extend AD to E such that EB is perpendicular AB. given that BE = 3, BA = 4, find the integer nearest to BC^2

Ans. **29**

Sol. In Δ ABD -



$$\tan \alpha = \frac{3}{4}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan 2\alpha = \frac{24}{7}$$

In Δ ABC -

$(1 + 1) \cot \theta = (1) \cot \alpha - 1 \cot 2\alpha$ (m –n rule)

$$= \frac{4}{3} - \frac{7}{24} = \frac{25}{24} \Rightarrow \cot\theta = \frac{25}{48}$$

In Δ ABD -

$$\frac{AB}{\sin(\pi-\theta)} = \frac{BD}{\sin\alpha} \qquad \text{(Sine Rule)}$$

$$\Rightarrow \frac{4}{\sin\theta} = \frac{BD}{\sin\alpha} \Rightarrow BD = \frac{4\sin\alpha}{\sin\theta} = 4 \times \frac{3}{5} \text{ Cosec } \theta$$

$$\Rightarrow BD = \frac{12}{5} \text{ Cosec } \theta$$

$$BC = 2(BD) = \frac{24}{5} \text{ Cosec } \theta$$

$$(BC)^2 = \frac{24 \times 24}{25} \times \text{Cosec}^2\theta = \frac{24 \times 24}{24} \times \frac{(48^2 + 25^2)}{48 \times 48} = \frac{2929}{100} = 29.29$$

24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the numbers of ball in any two of the boxes is not a multiple of 6. If N = 100a + b, where a, b are positive integers less than 100, find a + b.

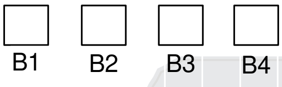
Ans. **81**

Sol. 52 identical balls

4 different Boxes. Such that no Box remains empty

Difference between the number of balls in any

Two Boxes not multiple of 6.



Boxes.

let a_1 goes in B_1 & So on

$a_1 + a_2 + a_3 + a_4 = 52$

give one ball to each box

clearly no two box will have same number of balls.

let $a_1 = 6q_i + r_i$

so $6q_1 + r_1 + 6q_2 + r_2 + \ldots\ldots + 6q_4 + r_4 = 48$

$6(q_1 + q_2 + \ldots. + q_4) + (r_1 + r_2 + r_3 + r_4) = 48$

Now we have only these cases

$r_1 + r_2 + r_3 + r_4 = 6$	&	$r_1 + r_2 + r_3 + r_4 = 12$
0 1 2 3		0 3 4 5
		1 32 4 5

so total $^{10}C_3 \times 4! + {}^{9}C_3 \times 2 \times 4!$

$= 24\ (120+168)$

$69 \times 100 + 12$

so Ans. = 69+1= 81

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TOTAL ENROLLED STUDENTS

(April 2001 to March 2022)

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Classroom: 5,44,210

ResoBASE: 70,844

DLP+eLP: 3,27,256

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