Resonance
Educating for better tomorrow

Regional Mathematical Olympiad-2014
क्षेत्रीय गणित ओलिंपियाड-2014

Time : 3 hours (समय: 3 घंटा) **December 07, 2014 (दिसम्बर 07, 2014)**

Instructions (निर्देश) :

- **Calculators (in any form) and protractors are not allowed.**
  किसी भी तरह के गुणक **(Calculators)** तथा चांदा के प्रयोग की अनुमति नहीं है।
- **Rulers and compasses are allowed.**
  पैमाना **(Rulers)** तथा परकार **(compasses)** के प्रयोग की अनुमति है।
- **Answer all the questions. All questions carry equal marks. Maximum marks : 102**
  सभी प्रश्नों के उत्तर दीजिये। सभी प्रश्नों के अंक समान हैं, अधिकतम् अंक : **102**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
  प्रत्येक प्रश्न का उत्तर नए पेज से प्रारंभ कीजिये। प्रश्न क्रमांक स्पष्ट रूप से इंगित कीजिये।

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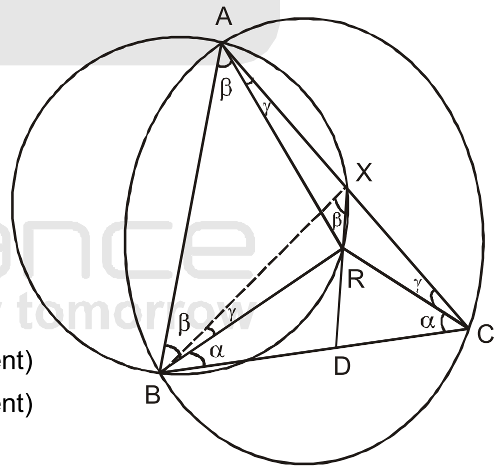
1. Let ABC be an acute-Angled triangle and suppose $\angle ABC$ is the largest angle of the triangle Let R be its circumcentre. Suppose the circumcircle of triangle ARB cuts AC again in X. Prove that RX is perpendicular to BC.

मान लीजिये कि ABC कए न्यून कोण त्रिभुज हैं तथा $\angle ABC$ त्रिभुज का सबसे बड़ा कोण है, मान लीजिए R इसका परिकेन्द्र है, यह भी मानिये कि त्रिभुज ARB का परिवृत्त AC को पुनः X पर काटता है, सिद्ध कीजिये कि RX, BC के लम्बवत् है।

Sol. AR = BR = CR ($\because$ circumciradius)

$\therefore \angle RBC = \angle RCB = \alpha$

$\therefore \angle RAB = \angle ABR = \beta$



$\therefore \angle RAC = \angle RCA = \gamma$

sum of angles in $\Delta ABC = 2(\alpha + \beta + \gamma) = 180^\circ$

$\therefore \alpha + \beta + \gamma = 90^\circ$ ..... (i)

$\angle RBX = \angle RAX = \gamma$ ($\because$ angles in same segment)

$\angle BAR = \angle BXR = \beta$ ($\because$ angles in same segment)

$\therefore \Delta BXD$,

$\alpha + \beta + \gamma + \angle XDB = 180^\circ$

$\angle XDB = 90^\circ$ **(use equation (i))**

$\therefore DX \perp BC$.

2. Find all real numbers x and y such that
वह सभी वास्तविक संख्याएं x तथा y ज्ञात कीजिये जिनके लिए

$x^2 + 2y^2 + \frac{1}{2} \le x(2y+1)$

Sol. $x^2 + 2y^2 + \frac{1}{2} \le x(2y + 1)$

$2x^2 + 4y^2 + 1 \le 4xy + 2x$

$2x^2 + 4y^2 - 2x - 4xy + 1 \le 0$

$x^2 + x^2 + 4y^2 - 2x - 4xy + 1 \le 0$

$(x^2 + 1 - 2x) + (x^2 + 4y^2 - 4xy) \le 0$

$(x - 1)^2 + (x - 2y)^2 \le 0$

so sum of two square can not be negative.

$\therefore (x - 1)^2 + (x - 2y)^2 = 0$

$\therefore x - 1 = 0$ and $x - 2y = 0$

$x = 1$ and $x = 2y$

$\therefore x = 1$

$y = x/2 = \frac{1}{2}$

3. Prove that there does not exist any positive integer n < 2310 such that n (2310 –n) is a multiple fo 2310.
सिद्ध कीजिये कि ऐसे कोई धनात्मक पूर्णांक n < 2310 का आस्तित्व नहीं है जिसके लिए 2310 का एक गुणक n(2310 –n) है।

Sol. As n (2310 –n) is a multiple fo 2310

$n(2310 - n) = 2310k$ (where k is some integer)

$2310n - n^2 = 2310k$

$2310n - 2310k = n^2$

$2310(n - k) = n^2$

$n - k = \frac{n^2}{2310}$

as $n - k$ is an integer, so $n^2/2310$ is also an integer

$\therefore n^2$ is divisible by 2310.

as $2310 = 2 \times 3 \times 5 \times 7 \times 11$

$\therefore$ n is also divisible by 2310

but n is less than 2310 n < 2310 so

we can say that n is not divisible by 2310.

4. Find all positive real numbers x, y, z such that
वह सारी धनात्मक संख्याएँ x, y, z ज्ञात कीजिये जिसके लिए

$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2014}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2014}$

Sol. $2x - 2y + \frac{1}{z} = \frac{1}{2014}$ ...(1)

$2y - 2z + \frac{1}{x} = \frac{1}{2014}$ ...(2)

$2z - 2x + \frac{1}{y} = \frac{1}{2014}$ ...(3)

by adding (1), (2), (3) we get

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}$ ...(4)

from (1) $2xz - 2yz + 1 = \frac{z}{2014}$ ...(5)

from (2) $2xy - 2zx + 1 = \frac{x}{2014}$ ...(6)

from (3) $2zy - 2xy + 1 = \frac{y}{2014}$ ...(7)

by adding (5), (6), (7) we get

$$3 = \frac{x+y+z}{2014}$$

$x + y + z = 2014 \times 3$ ...(8)

(4) $\times$ (8)

$$(x + y + z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 9$$

It is possible only when

$x = y = z$

$\therefore$ because AM, HM for x, y, z

AM $\geq$ HM

$$\frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$(x + y + z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

$$(x + y + z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 9$$

if only when term are equal

Put x = y = z = a in (1)

$$2a - 2a + \frac{1}{a} = \frac{1}{2014}$$

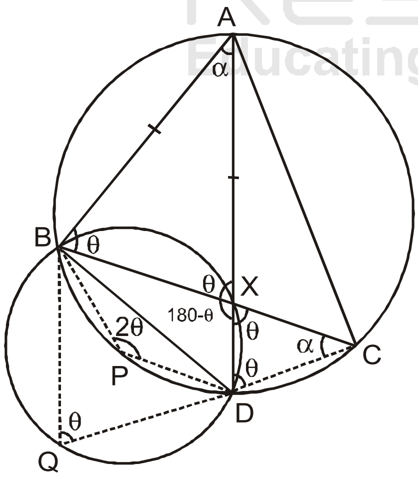
a = 2014

$\therefore$ x = y = z = 2014

5. Let ABC be a triangle. Let X be on the segment BC such that AB = AX. Let AX meet the circumcircle Γ of triangle ABC again at D. Show that the circumcentre of ΔBDX lies on Γ.

मान लीजिये की ABC एक त्रिभुज है। मान लीजिये कि BC का एक अनुभाग X इस प्रकार हैं कि AB = AX मान लीजिये कि AX त्रिभुज ABC के परिवत्त Γ पर पुनः D. पर मिलती है। दिखाईये कि ΔBDX का परिकेन्द्र Γ की परिधि पर निहित है।

Sol.



Given : A triangle ABC with a point x on BC such that AB = AX and AX produced meets circumcircle Γ of ΔABC at D.

Construction : Join D & C. Let P be a point on $\overset{\frown}{BD}$ Join P to B & D . Circumcircle of ΔBDX is drawn and Q be a point on it. Join Q to B & D

To prove : Circumcentre of ΔBDX lies on circumcircle Γ of ΔABC.

Proof : Let $\angle ABX = \theta = \angle ABC$

and $\angle BAX = \alpha = \angle BAD$

$\angle ADC = \angle ABC = \theta$ ($\because$ angle in same segment)

$\angle BCD = \angle BAD = \alpha$ ($\because$ angle in same segment)

Now in ΔCDX

$\angle CDX + \angle DXC + \angle XCD = 180°$ ($\because$ sum of interior angles of a triangle is 180°)

$\Rightarrow \theta + \theta + \alpha = 180°$

$\Rightarrow 2\theta = 180 - \alpha$ ..(1)

Now ABPD is a cyclic quadrilateral

so $\angle BAD + \angle BPD = 180°$ ($\because$ sum of opposite angles of a cyclic quadralitral is 180°)

$\Rightarrow \angle BPD = 180 - \alpha$

From eq. (1) $\angle BPD = 2\theta$ ..(2)

Now BXDQ is also a cyclic quadrilateral

so $\angle BQD + \angle BXD = 180°$ ($\because$ sum of opposite angles of a cyclic quadralitral is 180°)

$\Rightarrow \angle BQD + 180 - \theta = 180°$ ($\because \angle BXD = 180° - \angle BXA = 180° - \theta$)

$\Rightarrow \angle BQD = \theta$ ..(3)

($\because \angle BXD = 180° - \angle BXA = 180° - \theta$)

By eq. (2) & (3) $\angle BPD = 2\angle BQD$

That means P is centre of circumcircle of ΔBDX ($\because$ angle made by arc on circumfereance is half of angle made on centre in a circle)

so circumcentre of DBDX lies on circumcircle Γ of ΔABC **Hence Proved.**

6. For any natural number n, let S(n) denote the sum of the digits of n. Find the number of all 3-digit numbers n such that S(S(n)) = 2.

किसी प्राकृत संख्या n, के लिए, मान लीजिये कि S(n) के अंकों के योग को प्रकट करता है, उन सभी तीन अंकों की संख्या n की कुल संख्या ज्ञात कीजिये जिनके लिए S(S(n)) = 2.

Sol. Let no. be abc

Let a + b + c = xy

given x + y = 2

1st case : x = 0, y = 2

2nd case : x = 1, y = 1

3rd case : x = 2, y = 0

1st case :

a + b + c = 0

a $\neq$ 0 if a = 1

b or c = 0

101, 110 ..(1)

If a = 2

no. = 200 ..(2)

2nd case

a + b + c = 11

If a = 1, b + c = 10 9 case

a = 2, b + c = 9 10 case

a = 3 b + c = 8 9 case

a = 4 b + c = 7 8 case

a = 5 b + c = 6 7 case

a = 6 b + c = 5 6 case

a = 7	b + c = 4	5 case
a = 8	b + c = 3	4 case
a = 9	b + c = 2	3 case
Total =		61 case

3rd case

a + b + c = 20

a $\neq$ 1 because b + c can be greater than 18

a = 2, b + c = 18	1 case
a = 3, b + c = 17	2 case
a = 4, b + c = 16	3 case
a = 5, b + c = 15	4 case
a = 6, b + c = 14	5 case
a = 7, b + c = 13	6 case
a = 8, b + c = 12	7 case
a = 9, b + c = 11	8 case
Total =	36 case

Total = 36 + 61 + 3 = 100 cases