KAPREKAR CONTEST - FINAL - SUB JUNIOR

Classes VII & VIII

AMTI - Saturday, 2nd November_2019.

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

---

1. Let a_n be the units place of $1^2 + 2^2 + 3^2 + \ldots + n^2$. Prove that the decimal $0.\,a_1a_2a_3\ldots a_n\ldots$ is a rational number and represent it as $\frac{p}{q}$, where p and q are natural numbers.

Sol. For $1^2 + 2^2 + 3^2 + \ldots + n^2$

$a_1 = 1$	$a_{11} = 6$	$a_{21} = 1$
$a_2 = 5$	$a_{12} = 0$	$a_{22} = 5$
$a_3 = 4$	$a_{13} = 9$	$a_{23} = 4$
$a_4 = 0$	$a_{14} = 5$	$a_{24} = 0$
$a_5 = 5$	$a_{15} = 0$	$a_{25} = 5$
$a_6 = 1$	$a_{16} = 6$	$a_{26} = 1$
$a_7 = 0$	$a_{17} = 5$	$a_{27} = 0$
$a_8 = 4$	$a_{18} = 9$	$a_{28} = 4$
$a_9 = 5$	$a_{19} = 0$	$a_{29} = 5$
$a_{10} = 5$	$a_{20} = 0$	$a_{30} = 5$

$\therefore$ given number $0.\,a_1a_2a_3\ldots a_n = 0.\overline{15405104556095065900}$

$\therefore$ given number is non-terminating and repeating.

$\therefore$ it is a rational number and can be represent in the form of $\frac{p}{q}$.

2. **(a)** Find the positive integers m, n such that $\frac{1}{m} + \frac{1}{n} = \frac{3}{17}$.

(b) Find the positive integers m, n, p such that $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{3}{17}$.

(c) Using this idea, prove that we can find for any positive integer k, k distinct integers, $n_1, n_2, \ldots, n_k$ such that $\frac{1}{n_1} + \frac{1}{n_2} + \ldots \frac{1}{n_k} = \frac{3}{17}$.

Sol.

(a) We know

If $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$

then, $(x - a)(y - a) = a^2$

so, $\frac{1}{m}+\frac{1}{n}=\frac{3}{17}$

$\frac{1}{3m}+\frac{1}{3n}=\frac{1}{17}$

(3m – 17) (3n – 17) = 289

= 289 × 1

= 17 × 17

= 1 × 289

If (3m – 17) = 289 and 3n – 17 = 1

m = 102 n = 6

so, (m, n) = (102, 6)

If 3m – 17 = 17 and 3n – 17 = 17

$m=\frac{34}{3}$ $n=\frac{34}{3}$

not integer so reject

If (3m – 17) = 1 and 3n – 17 = 289

m = 6 n = 102

so, (m, n) = (6, 102)

so, $\frac{1}{6}+\frac{1}{102}=\frac{3}{17}$ ....(i)

(b) Now,

If $\frac{1}{6}=\frac{1}{x}+\frac{1}{y}$

(x – 6) (y – 6) = 36

= 1 × 36 or 36 × 1

= 2 × 18 or 18 × 2

= 3 × 12 or 12 × 3

= 4 × 9 or 9 × 4

= 6 × 6

so, (x, y) = (7,42) (8, 24), (9, 18), (10, 15) (12, 12)

so from equation (i)

$\frac{1}{7}+\frac{1}{42}+\frac{1}{102}=\frac{3}{17}$

$\frac{1}{8}+\frac{1}{24}+\frac{1}{102}=\frac{3}{17}$

: :

: :

and $\frac{1}{102}=\frac{1}{w}+\frac{1}{z}$

(w – 102) (z – 102) = $(102)^2$

= 1 × 10404

= 2 × 5202

= : :

= : :

= 102 × 102

Total 27 in which 13 are repeated so total 14 different pais.

so pairs of (w, z) = (103, 10506), (104, 5304).......... (204, 204)

so total 14 pairs

from equation (i)

$$\frac{1}{6}+\frac{1}{103}+\frac{1}{10506} = \frac{3}{17}$$

$$\frac{1}{6}+\frac{1}{104}+\frac{1}{5304} = \frac{3}{17}$$

: :

: :

Total 5 + 14 = 19 pairs.

(c) $\because \quad \frac{1}{a}=\frac{1}{a+1}+\frac{1}{a(a+1)}$

We convert every rational number into definite unit fractions so we can find for any positive integer k.

Such that $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+......+\frac{1}{n_k}=\frac{3}{17}$.

3. Does there exist a positive integer which is a multiple of 2019 and whose sum of the digits is 2019? If no, prove it. If yes, give one such number.

Sol.

	Sum of digits
1 × 2019 = 2019	12
2 × 2019 = 4038	15
3 × 2019 = 6057	18
4 × 2019 = 8076	21
5 × 2019 = 10095	16
6 × 2019 = 12114	9

So, sum of digits of number 20196057 = 30

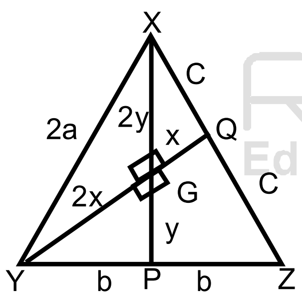
If we take 67 times 20196057 and 1 time 12114 then sum of digits is 2019 and number is also divisible by 2019.

Number is $\underbrace{20196057\ 20196057......}_{\text{67 times}}12114$

other number is $\underbrace{4038\ 4038.....}_{\text{134 times}}12114$.

4. In a triangle XYZ, the medians drawn through X and Y are perpendicular. Then show that XY is the smallest side of XYZ.

Sol.



$\because$ XP ⊥ YQ, XP and YQ intersect at G

Let XY = 2a

YZ = 2b

XZ = 2c & XG = 2y

GP = y and YG = 2x

GQ = x

In ΔXGQ

$c^2 = 4y^2 + x^2$

$c = \sqrt{4y^2 + x^2}$ ....(1)

In ΔYGP

$b^2 = 4x^2 + y^2$

$b = \sqrt{4x^2 + y^2}$ ....(2)

In ΔXGY

$4a^2 = 4x^2 + 4y^2$

$a^2 = x^2 + y^2$

$a = \sqrt{x^2 + y^2}$ .....(3)

from eq.(1) & (3)

$a < c \Rightarrow 2a < 2c \Rightarrow XY < XZ$ .....(4)

From eq. (2) & (3)

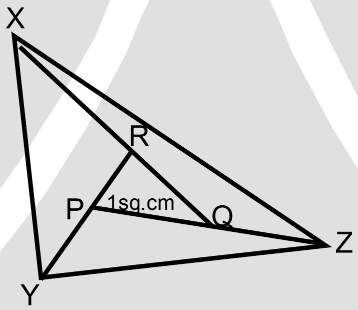
$a < b$

$2a < 2b$

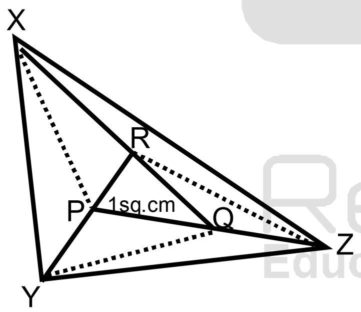
$\Rightarrow XY < YZ$ ....(5)

From eq. (4) & (5) we can say that XY is the smallest side.

5. Let ΔPQR be a triangle of area $1cm^2$. Extend QR to X such that QR = RX ; RP to Y such that RP = PY and PQ to Z such that PQ = QZ. Find the area of ΔXYZ.



Sol.



area ΔPRQ = area ΔPXR (PR is a median)

1 = area ΔPXR

area ΔPXR = area ΔPXY (PX is a median)

1 = area ΔPXY

area ΔPQY = area ΔPRQ (PQ is a median)

area ΔPQY = 1

area ΔQYZ = area ΔPQY (YQ is a median) area RQZ = area ΔPRQ (RQ is a median)

area ΔRQZ = 1

area ΔRQZ = area ΔRZX (RZ is a median)

1 = area ΔRZX

$\therefore$ area ΔXYZ = 7 cm^2

6. Find the real numbers x and y given that $x - y = \frac{3}{2}$ and $x^4 + y^4 = \frac{2657}{16}$.

Sol. $x^4 + y^4 = \frac{2657}{16}$

$(x^2)^2 + (y^2)^2 = \frac{2657}{16}$

$(x^2 + y^2)^2 - 2x^2y^2 = \frac{2657}{16}$

$(x^2 + y^2)^2 - 2(xy)^2 = \frac{2657}{16}$

$((x - y)^2 + 2xy)^2 - 2(xy)^2 = \frac{2657}{16}$

$xy = t$

$x - y = \frac{3}{2}$

$\left(\left(\frac{3}{2}\right)^2 + 2t\right)^2 - 2t^2 = \frac{2657}{16}$

$\frac{81}{16} + 4t^2 + 9t - 2t^2 = \frac{2657}{16}$

$2t^2 + 9t = \frac{2576}{16}$

$2t^2 + 9t - 161 = 0$

$2t^2 + 23t - 14t - 161 = 0$

$t(2t + 23) - 7(2t + 23) = 0$

$(2t + 23)(t - 7) = 0$

$t = 7$, $t = \frac{-23}{2}$

$xy = 7$ or $xy = \frac{-23}{2}$

when $xy = 7$

$y = \frac{7}{x}$

$x - y = \frac{3}{2}$

$x - \frac{7}{x} = \frac{3}{2}$

$\frac{x^2 - 7}{x} = \frac{3}{2}$

$2x^2 - 14 = 3x$

$2x^2 - 3x - 14 = 0$

$2x^2 - 7x + 4x - 14 = 0$

$x(2x - 7) + 2(2x - 7) = 0$

$(2x - 7)(x + 2) = 0$

$x = \frac{7}{2}$, $x = -2$

when $x = \frac{7}{2}$

$y = 2$

when $x = -2$

$y = \frac{-7}{2}$

when $xy = \frac{-23}{2}$

$y = \frac{-23}{2x}$

$x - y = \frac{3}{2}$

$\frac{x}{1} + \frac{23}{2x} = \frac{3}{2}$

$\frac{2x^2 + 23}{2x} = \frac{3}{2}$

$2x^2 - 3x + 23 = 0$

D = –ve

No real value

7. The difference of the eight digit number ABCDEFGH and the eight digit number GHEFCDAB is divisible by 481. Prove that C = E and D = F.

Sol. Difference of ABCDEFGH – GHEFCDAB is k.

k = 1000000 AB + 10000 CD + 100EF + GH – 1000000GH – 10000EF – 100CD – AB

k = 999999 (AB – GH) + 9900(CD – EF)

here 999999 is divisible by 481 so 9900(CD – EF) should be divisible by 481.

9900(10C + D – 10E – F) = 481x

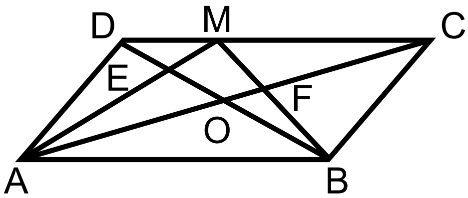
99000 (C – E) + 9900 (D – F) = 481 x

It is possible when C – E or D – F should be multiple of 37 or 0.

$\Rightarrow$ C – E = 0 , D – F = 0

C = E D = F

8. ABCD is a parallelogram with area $36cm^2$. O is the intersection point of the diagonals of the parallelogram. M is a point on DC. The intersection point of AM and BD is E and the intersection point of BM and AC is F. The sum of the areas of triangles AED and BFC is $12cm^2$. What is the area of the quadrilateral EOFM ?



Sol. area of parallelogram ABCD = 36

area ΔAOD = area ΔAOB = area ΔBOC = area ΔDOC = $\frac{1}{4}$ area ABCD = $\frac{1}{4}$ × 36 = 9

area ΔAMB = $\frac{1}{2}$ area ABCD = $\frac{1}{2}$ × 36 = 18

let area AED = x

$\therefore$ area ΔBFC = 12 – x

area BOF = area ΔBOC – area ΔBFC = 9 – (12 – x) = x – 3

area AOE = area ΔAOD – area ΔAED = 9 – x

area ΔAMB = area ΔAEO + area ΔAOB + area ΔBOF + area quadrilateral EOFM

18 = 9 – x + 9 + x – 3 + area quadrilateral EOFM or quadrilateral EOFM

$\Rightarrow$ area of quadrilateral EOFM = 3 cm^2.