NMTC STAGE-II 2015

SUB. JUNIOR GROUP (VII AND VIII)

Note : •

Answer as many questions as possible.
Elegant and noval solution will get extra credits.
Diagrams and explanations should be given wherever necessary.
Fill in FACE SLIP and your rough working should be in the answer book itself
Maximum time allowed is THREE hour.
All questions carry equal marks.

1.

	175										70	

The diagram above contains 13 boxes. The numbers in the second and the twelfth boxes are respectively 175 and 70. Fill up the boxes with natural numbers such that

(i) sum of all numbers in all the 13 boxes is 2015,

(ii) sum of the numbers in any three consecutive boxes is always the same.

The solution must contain the steps how you arrive at the numbers.

b) if x, y, z are real and unequal numbers, prove that
$2015x^2 + 2015y^2 + 6z^2 > 2\,(2012xy + 3yz + 3zx)$

Sol. (a)

a	175	b	a	175	b	a	175	b	a	175	b	a

$\therefore$ b = 70
$5a + 4b + 175 \times 4 = 2015$
$5a + 4 \times 70 + 700 = 2015$
$5a + 280 + 700 = 2015$
$5a = 2015 - 980$
$5a = 1035$
$a = 207$

(b) $2012x^2$, $2012y^2$
we apply AM $\geq$ GM

$$\frac{2012x^2 + 20112y^2}{2} \geq \sqrt{(2012x^2)(2012y^2)}$$

$2012x^2 + 2012y^2 \geq 2[2012xy]$ ...(i)

$3y^2$, $3z^2$ apply AM $\geq$ GM

$$\frac{3y^2 + 3z^2}{2} \geq \sqrt{3y^2\,3z^2}$$

$3y^2 + 3z^2 \geq 2[(3)yz]$ ...(ii)

$3x^2$, $3z^2$ apply AM $\geq$ GM

$$\frac{3x^2 + 3z^2}{2} \geq \sqrt{3x^2\,3z^2}$$

$3x^2 + 3z^2 \geq 2[(3)xz]$ ...(iii)

Add (i), (ii), (iii)
$2015x^2 + 2015y^2 + 6z^2 > 2\,(2012xy + 3yz + 3zx)$

2. If a,b,c are reals such that a + b = 4 and $2c^2 - ab = 4\sqrt{3}\,c - 10$, find the numerical values of a, b and c.

Sol. (i)... $a + b = 4 \rightarrow b = 4 - a$

(ii).. $2c^2 - ab = 4\sqrt{3}\,c - 10$

put b = 4 – a in (ii)

$2c^2 - a(4 - a) = 4\sqrt{3}\,c - 10$

$2c^2 - 4a + a^2 = 4\sqrt{3}\,c - 10$

$2c^2 - 4\sqrt{3}\,c + a^2 - 4a + 10 = 0$

$2[c^2 - 2\sqrt{3}\,c + 3 - 3] + [a^2 - 4a + 4 - 4] + 10 = 0$

$2[(c - \sqrt{3})^2 - 3] + [(a - 2)^2 - 4] + 10 = 0$

$2(c - \sqrt{3})^2 - 6 + (a - 2)^2 - 4 + 10 = 0$

$2(c - \sqrt{3})^2 + (a - 2)^2 = 0$

Sum of two square number is zero , it is possible when both square have value zero

$\therefore \quad c = \sqrt{3}$, $a = 2$

So $b = 4 - a = 4 - 2 = 2$

3. When $a = 2^{2014}$ and $b = 2^{2015}$, prove that

$$\left\{\frac{\frac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\frac{1}{b-a}-\frac{1}{a+b}}\right\} \div \left\{\frac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}\right\}$$ is divisible by 3

Sol. $a = 2^{2014}$ $b = 2^{2015} = 2.2^{2014} = 2a$

$$\left\{\frac{\frac{(a+2a)^2+(a-2a)^2}{2a-a}-(a+2a)}{\frac{1}{2a-a}-\frac{1}{a+2a}}\right\} \div \left\{\frac{(a+2a)^3+(2a-a)^3}{(a+2a)^2-(a-2a)^2}\right\}$$

$$\left\{\frac{\frac{9a^2+a^2}{2a-a}-3a}{\frac{1}{a}-\frac{1}{3a}}\right\} \div \left\{\frac{27a^3+a^3}{9a^2-a^2}\right\}$$

$$\left[\frac{10a-3a}{\frac{3-1}{3a}}\right] \div \left[\frac{28a^3}{8a^2}\right]$$

$$= 7a \times \frac{3a}{2} \div \frac{7a}{2} = 7a \times \frac{3a}{2} \times \frac{2}{7a} = 3a$$

4. Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram on to its diagonals are the vertices of a parallelogram.

Sol. By AAS $\Delta DPO \cong \Delta BRO$

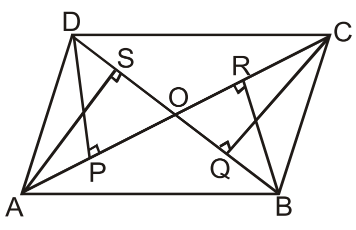
$\therefore$ PO = RO (CPCT)

By AAS $\Delta AOS \cong \Delta COQ$

$\therefore$ OS = OQ (CPCT)

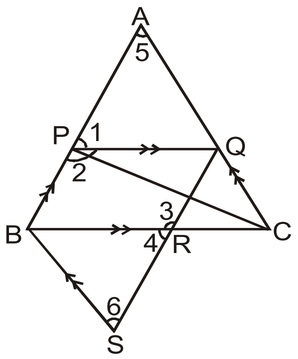
In quad PQRS diagonal bisect each other

$\therefore$ PQRS is ||gm



5. ABC is an acute angled triangle. P,Q are the points on AB and AC respectively such that area of Δ APC = area of ΔAQB. A line is drawn through B parallel to AC and meets the line trough Q parallel to AB at S. QS cuts BC at R. Prove that RS = AP.

Sol.



Area of ΔAPC = area ΔAQB
area ΔAPQ + area ΔPQC = area ΔAPQ + area ΔQPB
area ΔPQC = area ΔQPB
As ΔPQC ΔQPB have same area & same base, so they must lie between same parallel PQ || BC
As AQ || BS and AB || SQ
$\therefore$ ABSQ is ||gm
AQ = BS
$\angle 5 = \angle 6$
and PQ || BR and BP || QR
$\therefore$ PBRQ is ||gm
PQ = BR
$\angle 2 = \angle 3$
$\angle 2 = 180 - \angle 1$
$\therefore \angle 3 = 180 - \angle 1$
$\angle 4 = 180 - \angle 3 = 180 - (180 - \angle 1)$
$\angle 4 = \angle 1$
In ΔAQPand ΔSBR
AQ = BS
$\angle 4 = \angle 1$
$\angle 5 = \angle 6$
By AAS
$\Delta AQP \cong \Delta SBR$
AP = RS (CPCT)

6. (a) A man is walking from a town A to another town B at a speed of 4km/hr. He started one hour before a bus starts. The bus is travelling with a speed of 12kmlhr. The man on the way got into the bus and travels 2 hours and reached town B. What is the distance between town A and town B.

(b) A point P is taken within a rhombus ABCD such that PA=PC. Show that B,P,D are collinear.

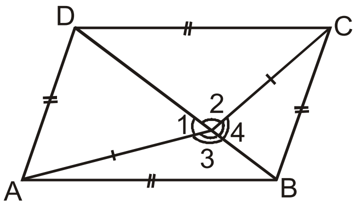
Sol.

Distance travelled by a man in 1 hr. = 4 × 1 = 4 km = AA'

Time required by bus to meet bus = $\dfrac{AA'}{\text{speed of bus} - \text{speed of man}} = \dfrac{4}{12-4}$ hr. = $\dfrac{1}{2}$ hr.

Let M be the meeting point of bus and man A'M = $4 \times \dfrac{1}{2}$ = 2 km

distance MB = 12 × 2 = 24 km
Total distance = AA' + A'M + MB = 4 + 2 + 24 = 30 km

(b) 

by SSS $\Delta APD \cong \Delta CPD$
$\angle 1 = \angle 2$ (CPCT)
By SSS $\Delta PAB \cong \Delta PCB$
$\angle 3 = \angle 4$ (CPCT)
$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$
$2\angle 1 + 2\angle 3 = 360°$
$\angle 1 + \angle 3 = 180°$
$\therefore$ B,P,D are collinear

7. If $(x + y + z)^3 = (y + z - x)^3 + (z + x - y)^3 + (x + y - z)^3 + kxyz$ find the numerical value of k. Deduce the following result.

If a= 2015, b = 2014, c = $\frac{1}{2014}$ prove that

$(a + b + c)^3 – (a + b - c^3) – (b + c – a)^3 –(c + a – b)^3 – 23abc = 2015$

Sol. $(x + y + x)^3 = (y + 2 – x)^3 + (2 + x – y)^3 + (x + y – 2)^3 + kxyz$

we see that by putting x = 0, the expression vanishes.

so x is a factor of the expression similarly y,z are factors of that exprssion.

The given expression is of 3rd degree & the factors so far obtained are also of 3rd degree hence if there is any constant factors supposing it to be k.

Then in order to find k

put x = 1, y = 1, z = 1

$3^3 – 1^3 – 1^3 – 1^3 = k .1.1.1$

k = 24

Hence 24 xyz

If a = 2015, b = 2014, c = $\frac{1}{2014}$

so value of $(a + b + c)^3 – (a + b – c)^3 – (b + c – a)^3 – (c + a – b)^3 – 23abc$

= 24 abc – 23abc

= abc $2015 \times 2014 \times \frac{1}{2014}$ = 2015