# NMTC STAGE-II 2015

## JUNIOR GROUP (IX AND X)

**Note : •**
Answer as many questions as possible.
Elegant and noval solution will get extra credits.
Diagrams and explanations should be given wherever necessary.
Fill in FACE SLIP and your rough working should be in the answer book itself
Maximum time allowed is THREE hour.
All questions carry equal marks.

**1.** a) 28 integers are chosen from the interval [104, 208]. Show that there exit two of them having a common prime divisor.

b) AB is a line segment.C is a point on AB. ACPQ and CBRS are squares drawn on the same side AB, Prove the S is the orthocentre of the triangle APB.

**Sol.** No. of primes between 104 to 208 $\Rightarrow$ 19
We have so many Integers which are $\rightarrow$
divisible by 2 $\longrightarrow$ 206 (2×103)................106(2×53)
divisible by 3 $\longrightarrow$ 183(3×61)................111(3×37)
divisible by 5 $\longrightarrow$ 105,110 ................205
divisible by 7 $\longrightarrow$ 105................203
divisible by 11 $\longrightarrow$ 110................198
divisible by 13 $\longrightarrow$ 169(13×13)................143(13×11)
divisible by 17 $\longrightarrow$ 119 (17×7)................187(17×11)
So that 19 primes and one from every 2,3,5,7,11 and 13 we have total $\Rightarrow$ 19 + 6 $\Rightarrow$ 25,
Such Numbers that doesn't have any common prime factor.
But, if we take 28 Numbers so their will be 2 such Numbers that will have 1 same prime factor.

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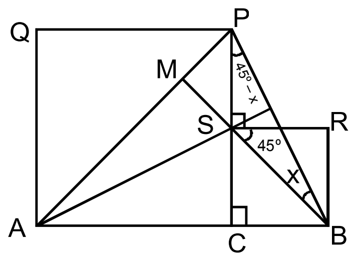
(b)




Draw a line from B which passes through S and intersect AP at M. As PC is the altitude of $\Delta$APB which pass through S and BM is also pass through S. In order to prove S orthocentre, now we just need to prove BM $\perp$ AP.

Let PBS = x

$\angle$SPB + $\angle$PSB + $\angle$PBS = 180º

$\angle$RSB = 45º [As BS is diagonal of the square]

$\rightarrow$ $\angle$SPB = 45º – x

In $\Delta$SPM

AP is the diagonal of square ACPQ

$\therefore$ $\angle$MPS = 45°

$\angle$PSM = 180 – (90 + 45) [Linear pair]

= 45°

$\therefore$ $\angle$PMS = 180 – (45 + 45) $\Rightarrow$ 90° [Angle sum proper]

In $\Delta$PAB

PC and BM are altituted which intrsecs at S.

So S is the orthocenter

**2.** a) a,b,c are distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the numericla value of abc.

b) $a = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ............ + \frac{1}{2015^2}$

find [a], where [a] denotes the integer part of a.

Sol. (a) $a^3 - b^3 = 3(b^2 - a^2) \Rightarrow a^2 + b^2 + ab = -3(a + b)$ ..(i)

$b^3 - c^3 = 3(c^2 - b^2) \Rightarrow b^2 + c^2 + bc = -3(c + b)$ ..(ii)
$a^3 - c^3 = 3(c^2 - a^2) \Rightarrow a^2 + c^2 + ac = -3(a + c)$ ..(iii)
from (i) & (ii)
$a^2 - c^2 + b(a - c) = -3(a - c)$
$a + b + c = -3$ ..(iv)
by adding given equation
$a^3 + b^3 + c^3 = 6(a^2 + b^2 + c^2) - 75$
$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc = 6(a^2 + b^2 + c^2) - 75$
$-3a^2 - 3b^2 - 3c^2 + 3ab + 3bc + 3ac + 3abc = 6(a^2 + b^2 + c^2) - 75$
$3ab + 3bc + 3ac + 3abc = 9(a^2 + b^2 + c^2) - 75$
$= 9[(a + b + c)^2 - 2ab - 2bc - 2ca)] - 75$
$= 9[9 - 2ab - 2bc - 2ac) - 75$
$3ab + 3bc + 3ac + 3abc = 6 - 18(ab + bc + ca)$
$3abc = 6 - 21(ab + bc + ca)$
$abc = 2 - 7(ab + bc + ca)$ ..(v)
from (i), (ii),(iii)
$2a^2 + 2b^2 + 2c^2 + ab + bc + ac = -3 \times 2 \times -3 = 18$
$2(a^2 + b^2 + c^2) + ab + bc + ca = 18$
$2((a + b + c)^2 - 2ab - 2bc - 2ca) + ab + bc + ca = 18$
$2(3)^2 - 3(ab + bc + ca) = 18$
$ab + bc + ca = 0$ ..(vi)
$\therefore$ from (v) & (vi)
$abc = 2 - 7(0) = 2$

(b) $\frac{1}{1 \times 2} > \frac{1}{2^2}$

$\frac{1}{2 \times 3} > \frac{1}{3^2}$

and so on ...

$\frac{1}{2014 \times 2015} > \frac{1}{2015^2}$

Add all above inequations

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + - + \frac{1}{2014 \times 2015} > \frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + - + \left(\frac{1}{2014} - \frac{1}{2015}\right) > \frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$$

$$\left(1-\frac{1}{2015}\right) > \frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$$

$$\left(\frac{2014}{2015}\right) > \frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$$

$$0.99950 > \frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$$

so it means $\frac{1}{2^2} + \frac{1}{3^2} + - + \frac{1}{2015^2}$ is a

decimal value less than one

$\therefore \quad [a] = \left[1+\frac{1}{2^2}+\frac{1}{3^2}+-+\frac{1}{2015^2}\right]$

= [1 + decimal value less than one]

= 1

**3.** The arthemetic mean of a number of pair wise distinct prime numbers is 27. Determine the biggest prime among them.

**Sol.** These are pairs of prime no. whose mean is 27

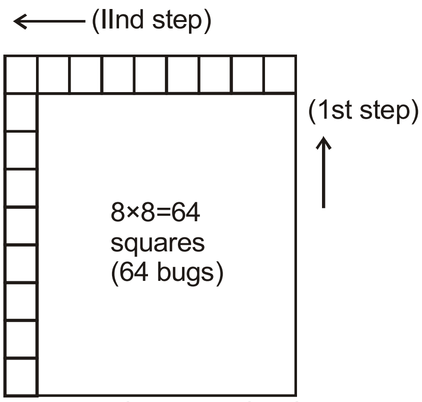
(47, 7) (11, 43) (41, 13) (19, 37)

So, biggest prime no. is 47

**4.** 65 bugs are placed at different squares of a 9 × 9 square board. A bug in each moves to a horizontal or vertical adjacent square. No bug makes two horizontal or two vertical moves in succession. Show that after some moves, there will be atleast two bugs in the same square.

Sol. Total bugs $\Rightarrow$ 65

total squares $\Rightarrow$ 81




If we take 64 bugs, then we can arrange them together into a matrix of 8 × 8 square, so their is a possibility that No 2 bugs are in same square, because we can move all the bugs vertically upward in 1st step, then Horizontally left in 2nd step vertically down in the third step, and in the 4th step horizontaly left and so on.

But , If we take 65 bugs so one horizontal or vertical row of square will fill with bugs. So we can not perform the above process in this situation [due to extra 65th bug]

so after some move their will be 2 bugs in same square.

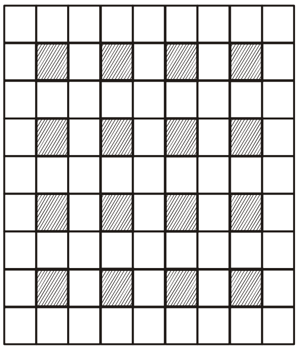
**"OR"**

We have consider 16 shaded squares.

Let we have a bug in the shaded Square. So in at most 4 moves, Bug will be in any shaded square again.

And if we have a bug in the un-shaded square, in at most 3 moves, bug will be in any shaded square again

So, if we have total 65 bugs in these 81 squares, some of them will be in shaded square and some of them in un-shaded square. So after 3 or 4 moves all the bugs need to be in shaded square. So their will exist atleast one move in which 2 bugs will get into the same shaded square-



**5.** f(x) is a fifth degree polynomial. It is given that f(x) + 1 in divisible by $(x-1)^3$ and f(x)–1 is divisible by $(x+1)^3$ . Find f(x).

Sol. Let $f(x) = k_1(x-\alpha)(x-\beta)(x-1)^3 - 1$

$f(x) - 1 = k_1(x-\alpha)(x-\beta)(x-1)^3 - 2$

$\therefore k_1(x-\alpha)(x-\beta)(x-1)^3 - 2 = k_2(x-\gamma)(x-\delta)(x+1)^3$

$\Rightarrow k_1(x^2 - (\alpha+\beta)x + \alpha\beta)(x^3 - 3x^2 + 3x - 1) - 2$

$= k_2(x^2 - (\gamma+\delta)x + \gamma\delta)(x^3 + 3x^2 + 3x + 1)$

comparing cofficient of $x^5$ $\quad k_1 = k_2 = k$ ..(i)

comparing cofficient of $x^4$

$-3k - k\alpha - k\beta = 3k - k\gamma - k\delta$

$\Rightarrow \quad \gamma + \delta - \alpha - \beta - 6 = 0$ ..(ii)

comparing cofficient of $x^3$

$3k + 3k\alpha + 3k\beta + k\alpha\beta = 3k - 3k\gamma - 3k\delta + k\gamma\delta$

$3\alpha + 3\beta + \alpha\beta + 3\gamma + 3\delta - \gamma\delta = 0$ ..(iii)

comparing cofficient of $x^2$

$-k - 3k\alpha - 3k\beta - 3k\alpha\beta = k - 3k\gamma - 3k\delta + 3k\gamma\delta$

$\Rightarrow \quad -1 - 3\alpha - 3\beta - 3\alpha\beta = 1 - 3\gamma - 3\delta + 3\gamma\delta$

$\Rightarrow \quad 3\gamma + 3\delta - 3\gamma\delta - 3\alpha - 3\beta - 3\alpha\beta - 2 = 0$ ..(iv)

comparing cofficient of x

$k\alpha + k\beta + 3k\alpha\beta = -k\gamma - k\delta + 3k\gamma\delta$

$\Rightarrow \quad \alpha + \beta + 3\alpha\beta + \gamma + \delta - 3\gamma\delta = 0$ ..(v)

comparing constant term :

$-k\alpha\beta - 2 = k\gamma\delta$

$\Rightarrow \quad k\alpha\beta + k\gamma\delta = -2$

$\Rightarrow \quad k(\alpha\beta + \gamma\delta) = -2$ ..(vi)

(v) – 3 × (iii) $\quad \alpha + \beta + 3\alpha\beta + \gamma + \delta - 3\gamma\delta = 0$

$9\alpha + 9\beta + 3\alpha\beta + 9\gamma + 9\delta - 3\gamma\delta = 0$

– – – – – +

$-8\alpha - 8\beta - 8\gamma - 8\delta = 0$

$\Rightarrow \quad \alpha + \beta + \gamma + \delta = 0$ ..(vii)

(ii) + (vii) $\gamma + \delta - \alpha - \beta = 6$

$\gamma + \delta + \alpha + \beta = 0$

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$2(\gamma + \delta) = 6$

$\Rightarrow \gamma + \delta = 3$ and $(\alpha + \beta) = (-3)$

put in (iv)

$3(3) - 3(-3) - 3(\alpha\beta + \gamma\delta) = 2$

$\Rightarrow 9 + 9 - 3\left(-\frac{2}{k}\right) = 2$ from (vi)

$18 + \frac{6}{k} = 2 \qquad \Rightarrow \frac{6}{k} = -16 \qquad k = -\frac{3}{8}$

$\therefore \alpha\beta + \gamma\delta = \frac{-2}{-3} \times 8 = \frac{16}{3}$

now put $\gamma + \delta = 3$ and $(\alpha + \beta) = (-3)$ in (v)

$-3 + 3 + 3(\alpha\beta - \gamma\delta) = 0 \qquad \Rightarrow \alpha\beta = \gamma\delta$

$\therefore \alpha\beta = \gamma\delta = \frac{8}{3}$

$\therefore f(x) = k(x - \alpha)(x - \beta)(x - 1)^3 - 1$

$= \left(\frac{-3}{8}\right)\left[x^2 - (-3)x + \frac{8}{3}\right](x - 1)^3 - 1$

$= \left(\frac{-3}{8}\right)\left[\frac{3x^2 + 9x + 8}{3}\right](x - 1)^3 - 1$

$= \frac{(-9x^2 - 27x - 24)(x^3 - 3x^2 + 3x - 1) - 24}{24}$

$= \frac{1}{24}(-9x^5 + 27x^4 - 27x^3 + 9x^2 - 27x^4 + 81x^3 - 81x^2 + 27x - 24x^3 + 72x^2 - 72x + 24 - 24)$

$= \frac{1}{24}[-9x^5 + 30x^3 - 45x]$

$= \frac{1}{8}[-3x^5 + 10x^3 - 15x]$

$\because f(x) + 1 = k(x - 1)^3$

put x = 1

$f(1) + 1 = 0$

$\Rightarrow f(1) = -1$

Verification :

$f(x) = \frac{1}{8}[-3x^5 + 10x^3 - 15x]$

put x = 1

RHS = $\frac{1}{8}$ [–3 + 10 – 15] = $\frac{-8}{8}$ = (–1)

**OR**

f(x) +1 is divisible by $(x - 1)^3$

$f(x) + 1 = (x - 1)^3 Q_1(x)$

$f'(x) = 3(x - 1)^2 Q_1(x) + (x - 1)^3 Q'_1(x)$

$= (x - 1)^2 [(3Q_1(x) + Q'_1(x)(x - 1)]$

so we can say f'(x) is a multiple of $(x - 1)^2$

f(x) –1 is divisible by $(x + 1)^3$

$f(x) - 1 = (x + 1)^3 Q_2(x)$

$f'(x) = 3(x + 1)^2 Q_2(x) + (x + 1)^3 Q'_2(x)$

$= (x + 1)^2 [(3Q_2(x) + Q'_2(x)(x + 1)]$

so we can say f'(x) is a multiple of $(x + 1)^2$

$f'(x) = \lambda (x^2 - 1)^2$

$f'(x) = \lambda (x^4 - 2x^2 + 1)$

$f(x) = \lambda \left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right) + C$

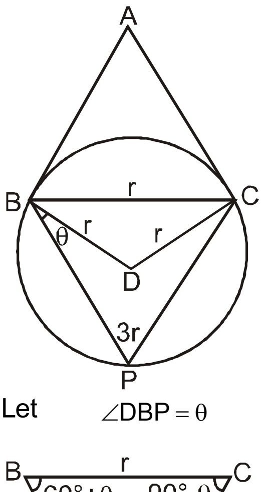
As f(1) = – 1 and f(–1) = 1

this gives C = 0 & $\lambda = -\frac{15}{8}$

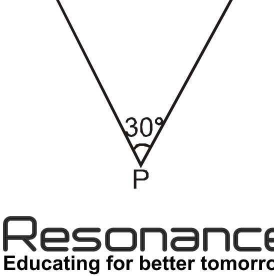
$f(x) = \frac{-3}{8} x^5 + \frac{5x^3}{4} - \frac{15}{8} x$

**6.** ABC and DBC are two equilateral triangles on the same base BC.A point P is taken on the circle with centre D, radius BD. Show that PA, PB, PC are the sides of a right triangle.

Sol.




Let $\angle DBP = \theta$

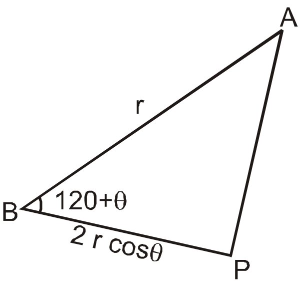


$$\frac{r}{\sin 30^\circ} = \frac{PC}{\sin(60+\theta)} = \frac{PB}{\sin(90-\theta)}$$

PB = 2r cosθ

PC = 2r sin (60+ θ)

PC = r $(\sqrt{3}\cos\theta + \sin\theta)$




$$\cos(120^\circ + \theta) = \frac{r^2 + 4r^2\cos^2\theta - AP^2}{2.r.2r\cos\theta}$$

$$-\sin(30^\circ + \theta) = \frac{r^2 + 4r^2\cos^2\theta - AP^2}{4r^2\cos\theta}$$

$AP^2 = r^2 + 4r^2\cos^2\theta + 4r^2\cos\theta\sin(30^\circ + \theta)$

$= r^2 [1+4\cos^2\theta + 4\cos\theta (\sin 30\cos\theta + \cos 30^\circ \sin\theta)$

$= r^2 [1+4\cos^2\theta + 4\cos\theta \frac{(\cos\theta + \sqrt{3}\sin\theta)}{2}$

$AP^2 = [1 + 4 \cos^2\theta + 2\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta]$

$AP^2 = [1 + 6 \cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta]$

Now, $PB^2 + PC^2$

$4r^2\cos^2\theta + r^2(\sqrt{3}\cos\theta + \sin\theta)^2$

$= r^2 [4\cos^2\theta + 3\cos^2 + \sin^2\theta + 2\sqrt{3} \sin\theta\cos\theta]$

$= r^2 [1 + 2\cos^2\theta + 4\cos^2\theta + 2\sqrt{3} \sin\theta\cos\theta]$

$AP^2 = [1 + 6 \cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta]$

$= AP^2$

**7.** a,b,c are real numbers such that a + b + c = 0 and $a^2 + b^2 + c^2 = 1$. Prove that $a^2b^2c^2 \le \frac{1}{54}$ .When does the equality hold ?

Sol. $(a - b)^2 \ge 0$

$a^2 + b^2 \ge 2ab$

$a^2 + b^2 + ab \ge 3ab$ .....(1)

a + b + c = 0

c = – a – b

$a^2 + b^2 + c^2 = a^2 + b^2 + (-a-b)^2 = 1$

$a^2 + b^2 + a^2 + b^2 + 2ab = 1$

$a^2 + b^2 + ab = \frac{1}{2}$ ...(2)

$a^2 + b^2 + ab \geq 3ab$

$\frac{1}{2} \geq 3ab$ ; $\frac{1}{6} \geq ab$

$ab \leq \frac{1}{6}$

$a^2b^2c^2 = a^2b^2(a^2 + b^2 + 2ab)$

$a^2b^2\left(\frac{1}{2} + ab\right)$

$a^2b^2\left(\frac{1}{2} + ab\right) \leq \frac{1}{36}\left(\frac{1}{2} + \frac{1}{6}\right) \leq \frac{1}{36} \times \frac{4}{6}$

$a^2b^2\left(\frac{1}{2} + ab\right) \leq \frac{1}{54}$ ; $a^2b^2c^2 \leq \frac{1}{54}$

Equality holds when $a = b = \frac{1}{\sqrt{6}}$ and $C^2 = 1 - a^2 - b^2$

$C \Rightarrow \sqrt{\frac{2}{3}}$

then $a^2b^2c^2 = \frac{1}{6} \times \frac{1}{6} \times \frac{2}{3} = \frac{1}{54}$